Portland State University

W’21 CS 584/684

Algorithm Design & Analysis

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Lecture 20

- Approx./R. algorithms
- Review

Randomized quicksort

- Pick the pivot randomly

```
Rand-QuickSort(A):
 if (array A has zero or one element)
    Return
 Pick pivot p ∈ A uniformly at random
 (L, M, R) ← PARTITION − 3 − WAY(A, p)      → O(n)
 Rand-QuickSort(L)                          → T(i)
 Rand-QuickSort(R)                          → T(n − i − 1)
```

Theorem. The expected number of compares to quicksort an array of n distinct elements is $O(n\log n)$.

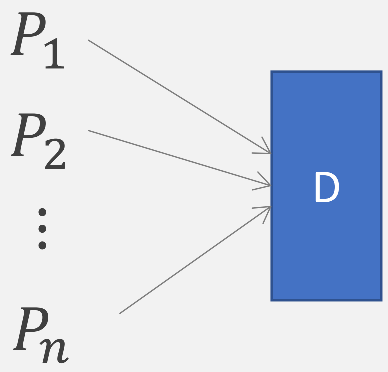
Contention resolution in a distributed system

Given: processes $P_1, \ldots, P_n$,

- each process competes for access to a shared database.
- If ≥ 2 processes access the database simultaneously, all processes are locked out.

Goal: a protocol so all processes get through on a regular basis

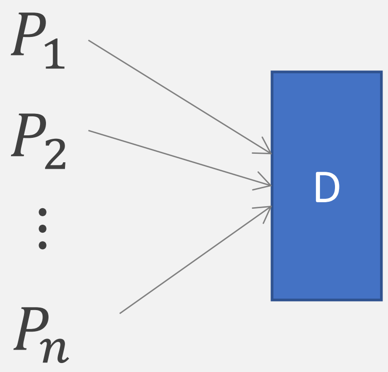
- Restriction: Processes can't communicate.



Contention resolution: randomized protocol

Protocol. Each process requests access to the database in round t with probability $p = 1/n$.

Theorem. All processes will succeed in accessing the database *at least once* within $O(n \ln n)$ rounds except with probability $\leq \frac{1}{n}$.



Randomized contention resolution: analysis 1

Def. $S[i,t]$ = event that process i succeeds in accessing the database in round t.

- Claim1. $\frac{1}{e \cdot n} \leq \Pr(S[i,t]) \leq \frac{1}{2n}$
- Pf. $\Pr(S[i,t]) = p(1-p)^{n-1}$

[Geometric distribution: independent Bernoulli trials]

Process i requests access   None of remaining request access

$\Rightarrow \Pr(S[i,t]) = \frac{1}{n}(1-1/n)^{n-1} \in [\frac{1}{en}, \frac{1}{2n}]$ $[p = 1/n]$

- $(1 - 1/n)^n$ converges monotonically from $1/4$ up to $1/e$.
- $(1 - 1/n)^{n-1}$ converges monotonically from $1/2$ down to $1/e$.

Randomized contention resolution: analysis 2

- Claim2. The probability that process i fails to access the database in $e \cdot n$ rounds is at most $1/e$. After $e \cdot n\ (c \ln n)$ rounds, the probability $\leq\ n^{-c}$.

- Pf. Let $F[i,t]$ = event that process i fails to access database in rounds 1 through t.

$$\Pr(F[i,t]) = \Pr\left(\overline{S[i,1]}\right) \cdot \ldots \cdot \Pr\left(\overline{S[i,t]}\right) \leq \left(1 - \frac{1}{en}\right)^t$$ [Independence & Claim 1]

  - Choose $t = en$: $\Pr(F[i,t]) \leq \left(1 - \frac{1}{en}\right)^{en} \leq \frac{1}{e}$
  - Choose $t = en \cdot c\ln n$: $\Pr(F[i,t]) \leq \left(\frac{1}{e}\right)^{c\ln n} \leq n^{-c}$

Randomized contention resolution: analysis 3

Theorem. All processes will succeed in accessing the database *at least once* within $2en \ln n$ rounds except with probability $\leq \frac{1}{n}$.

- **Pf.** Let $F[t]$ = event that some process fails to access database in rounds 1 through t.

Union Bound

Let E, F be two events. Then $\Pr(E \cup F) \leq \Pr(E) + \Pr(F)$.

$$\Pr(F[t]) = \Pr(\cup_{i=1}^{n} F[i,t]) \leq \sum_{i=1}^{n} \Pr(F[i,t]) \leq n \cdot \Pr(F[1,t])$$

- Choose $t = en \cdot 2\ln n$: $\Pr(F[t]) \leq n \cdot n^{-2} = 1/n$

Integer linear programming (ILP)

Input. Graph $G = (V, E)$

- Vertex cover $S \subseteq V$: subset of vertices that touches all edges

Goal. Find a vertex cover S of minimum size

- Formulating vertex cover as an integral linear program

For each $i \in V$, introduce $x_i \in \{0,1\}$
Min $\sum_{i=1}^{n} x_i$
Subject to:
$$x_i + x_j \geq 1 \quad \text{for each } (i,j) \in E$$

[i.e., Pick i in vertex cover iff. $x_i = 1$]

☹ We don't know (expect) a poly-time algorithm (ILP)

- Without integrality (LP), we do know poly-time algorithms

Putting aside the integral constraint

(ILP Π) Min $\sum_{i=1}^{n} x_i$
Subject to:

$$x_i + x_j \geq 1, \qquad \forall (i,j) \in E$$
$$x_i \in \{0,1\}, \qquad \forall i \in V$$

(LP Σ) Min $\sum_{i=1}^{n} x_i$
Subject to:

$$x_i + x_j \geq 1, \qquad \forall (i,j) \in E$$
$$0 \leq x_i \leq 1, \qquad \forall i \in V$$

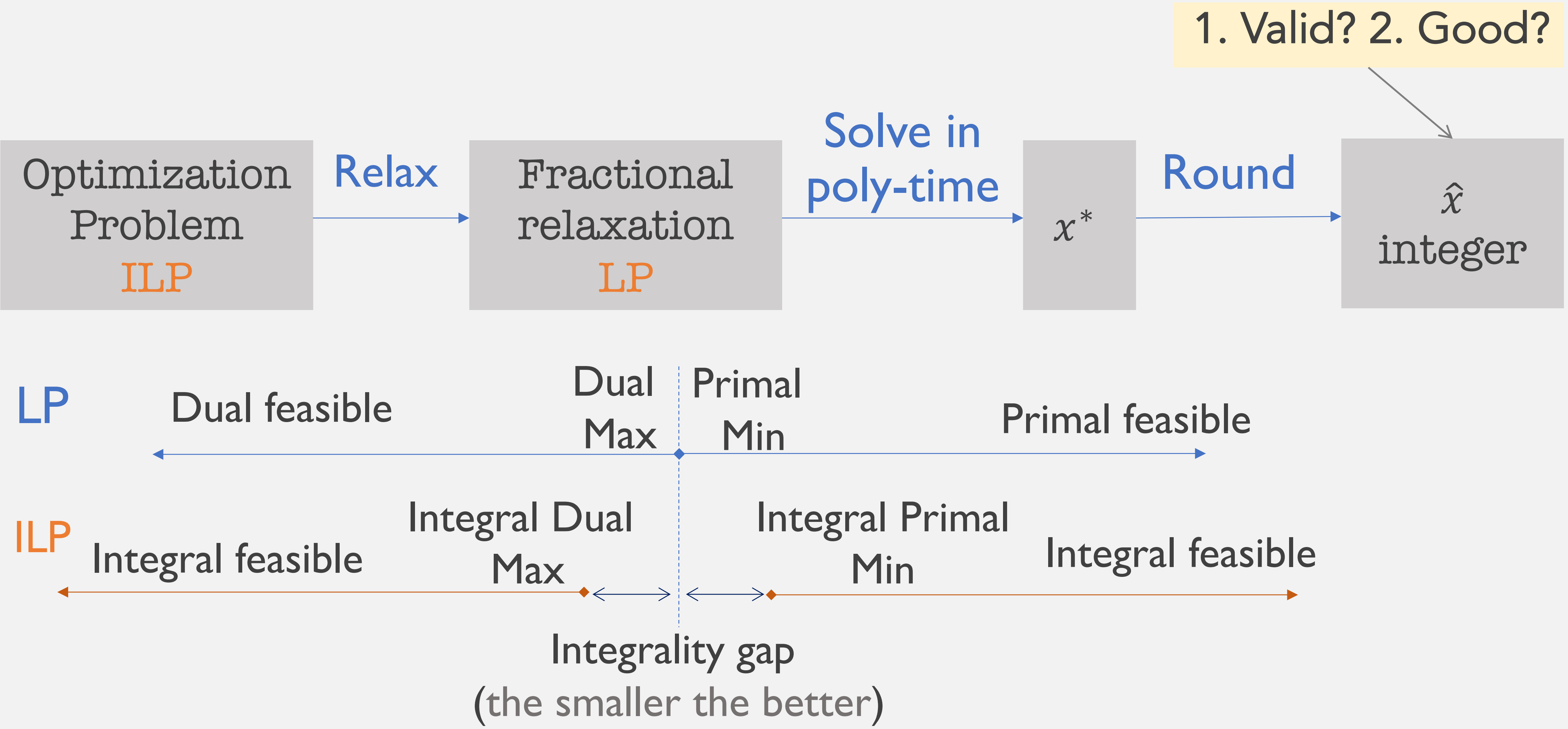
Let x^* be an optimal soln. for LP Σ
& optimal value $\mathrm{OPT} = \sum_i x_i^*$

?

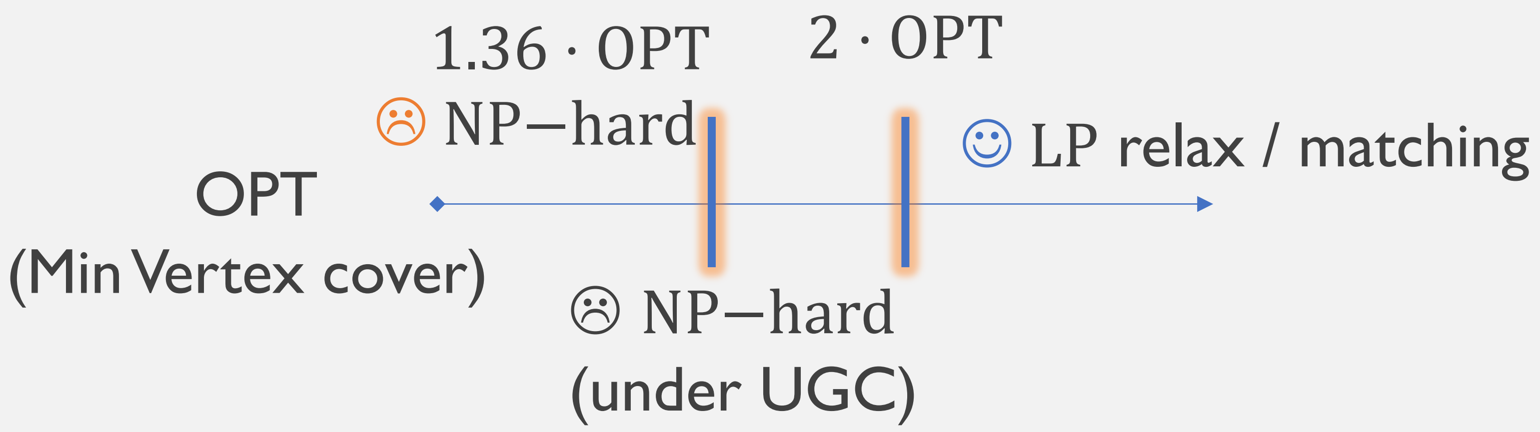
$$x_i := \lfloor x_i^* \rceil = \begin{cases} 1, & \text{if } x_i^* \geq \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

- (Threshold) Rounding:
  i. $\{x_i\}$ is a feasible integral solution: $\forall (i,j) \in E$, $x_i^* \geq \frac{1}{2}$ or $x_j^* \geq \frac{1}{2}$ or both
  ii. $\sum_i x_i \leq \sum_i 2 \cdot x_i^* = 2 \cdot \mathrm{OPT} \leq 2 \cdot \mathrm{OPT}_{\mathrm{Int}}$ [optimal value of ILP Π, i.e. size of min vertex cover]

LP relaxation



Hardness of approximation



Theorem. It is NP-Hard to approximate Vertex Cover to with any factor below 1.36067. [i.e., otherwise, you can solve 3-SAT in poly-time]

Theorem'. It is NP-Hard to approximate Vertex Cover to with any factor below 2, assuming the unique games conjecture (UGC).

Want to read more?

https://cs.nyu.edu/~khot/papers/UGCSurvey.pdf

https://cs.stanford.edu/people/trevisan/pubs/inapprox.pdf

Approximating set cover

Input. Set U of n elements, $S_1, \dots, S_m$ of subsets of U

Goal. Find $I \subseteq \{1, \dots, m\}$ of minimum size such that $\bigcup_{i \in I} S_i = U$

(ILP Π for Set cover)

For each $i \in \{1, \dots, m\}$, introduce $x_i \in \{0,1\}$

Min $\sum_{i=1}^{m} x_i$

Subject to:

$$\sum_{i: u \in S_i} x_i \geq 1, \qquad \forall u \in U$$

LP relaxation for set cover

(Set cover ILP Π)
Min $\sum_{i=1}^{m} x_i$
Subject to:
$$\sum_{i:u\in S_i} x_i \geq 1, \qquad \forall u \in U$$
$$x_i \in \{0,1\}, \qquad \forall i \in \{1,\dots,m\}$$

(Set cover Σ)
Min $\sum_{i=1}^{m} x_i$
Subject to:
$$\sum_{i:u\in S_i} x_i \geq 1, \qquad \forall u \in U$$
$$0 \leq x_i \leq 1, \forall i \in \{1,\dots,m\}$$

Let x^* be an optimal soln. for LP Σ & optimal value $\mathrm{OPT} = \sum_i x_i^*$

? $x_i := \lfloor x_i^* \rceil$

- Threshold rounding: does it cover all elements?
  - Ex. $u \in S_1, \dots, S_{100}; x_1^*, \dots x_{100}^* = \frac{1}{100} \Rightarrow x_1 = \cdots = x_{100} = 0$. u is missed!
- Randomized rounding!

LP relaxation for set cover

(Set cover ILP Π) Min $\sum_{i=1}^{m} x_i$
Subject to:
$$\sum_{i:u\in S_i} x_i \geq 1,\ \forall u \in U$$
$$x_i \in \{0,1\},\qquad \forall i \in \{1,\ldots,m\}$$

(Set cover LP Σ) Min $\sum_{i=1}^{m} x_i$
Subject to:
$$\sum_{i:u\in S_i} x_i \geq 1,\ \forall u \in U$$
$$0 \leq x_i \leq 1, \forall i \in \{1,\ldots,m\}$$

Let x^* be an optimal soln. for LP Σ
& optimal value $\mathrm{OPT} = \sum_i x_i^*$

☹ $x_i \coloneqq \lfloor x_i^* \rfloor$

- Randomized rounding: set $x_i = 1$ with probability x_i^*

$$\mathbb{E}[\sum_{i=1}^{m} x_i] = \sum_{i=1}^{m} \mathbb{E}[x_i] = \sum_{i=1}^{m} x_i{}^*$$

- But is it feasible? [Further analysis on Panigrahi's notes]

Theorem. There is a poly-time randomized algorithm achieving $O(\log n)$ expected approximation ratio, except w. probability $O(1/n)$.

You've accomplished a lot!

Be proud of yourselves!

Final exam

- When
  - Take-home. Release on Tuesday (03/16)at 4pm, due on Wednesday(03/17) at 4pm.
  - I will be online (slack) during 5:30 – 7:20pm to answer clarification questions.
- What
  - Comprehensive, slightly more focused on 2nd half.
- How
  - Similar format as mid-term: short-answer questions and algorithm designs.
  - No external resource permitted. Violations will be taken seriously.
  - No credit for unintelligible hand writing.

Principal questions

0. What problem to solve?

   be as precise as possible

1. Is the algorithm correct?

2. How much resource it costs?

   (time, space, …)

3. Can we do better?

DESIGN

ANALYSIS

Major topics

- Basics: asymptotic, graphs (BFS/DFS), data structures
- Algorithmic techniques
  1. Divide-&-Conquer
  2. Dynamic Programming
  3. Greedy
  4. Network flow & linear programming
  5. Randomization
  ★ Reduction
- Computational intractability: P, NP, NPC, approximation

1st half

2nd half

1. Divide-&-Conquer

- Idea
  - Divide into independent subproblems – recurse - combine
- Examples
  - Merge sort $O(n \log n)$
  - Fast multiplication $O(n^{1.59})$ [Karatsuba60]; $O(n \log n)$ [HarveyHoeven19]
  - Matrix multiplication $O(n^{2.81})$ [Strassen69]; $O(n^{2.376})$ [CoppersmithWinograd90];
  - Exponentiation $O(n \log n)$
  - Quick sort $O(n^2)$ worst-case; Expected $O(n \log n)$ random pivoting
- Analysis.
  - Solving recurrence: $T(n) = aT(n/b) + f(n)$
  - Recursion tree & Master theorem

2. Dynamic programming

- Idea
  - Divide into overlapping subproblems – smart recurse by memoization
  - Usually bottom-up iteration (topological order of implicit DAG)
- Examples
  - Fibonacci $O(n)$
  - Longest increasing subsequence $O(n^2)$
  - Weighted interval scheduling $O(n \log n)$
  - Matrix-chain multiplication $O(n^3)$
  - Longest common subsequence (aka Edit Distance) $O(mn)$
  - Shortest path (w. negative lengths) $O(mn)$ [Bellman-Ford]

3. Greedy

- Idea
  - Special case of DP: when lucky, lazy choice works
- Examples
  - Shortest path (w. non-negative lengths) $O((m+n)\log n)$ [Dijkstra]
  - Interval scheduling (weight = 1) $O(n \log n)$
  - Interval partitioning $O(n \log n)$
  - Minimum spanning tree $O(m \log n)$ [Kruskal]; $O((m+n)\log n)$ [Prim]
- Warning! 0 credit in exam without correctness proofs

Detour

- data structures [Prioirity Queue, Union Find]
- amortized analysis

4. Network flow - Linear programming

Network flow $\leq$ Linear programming

- Analytical
  - Max-Flow $\equiv$ Min-Cut
- Algorithms
  - Augmenting path: $O(mnC)$ [Ford-Fulkerson]
  - Capacity scaling: $O(m^2 \log C)$
  - In exam: quote $O(mn)$
- Applications
  - Bipartite perfect matching

- Analytical
  - Duality: OPT(Primal) = OPT(Dual)
- Algorithms
  - Simplex [efficient in practice/ but not poly-time worst-case]
  - Ellipsoid [poly-time but not practical]
  - Interior point [poly-time & practical]
- Warning: don't reduce to LP unless stated explicitly

5. Randomization

- Idea
  - Make random choices to get correct answers with high probability in (expected) poly-time.
- Examples
  - Contention resolution
  - Randomized quicksort
  - Randomized rounding for LP relaxation
- Important probabilistic tools
  - Union bound
  - Linearity of expectation

Computational intractability

- Classify problems by "hardness"
  - **P**: feasible problems (solvable in poly-time).
  - **NP**: ∃ poly-time certifier verifying a solution.

P vs. NP?

- Reduction: relating hardness ($A \leq B \Rightarrow A$ no harder than B)
  - Cook reduction [aka poly-time reduction]
  - Karp reduction [aka poly-time transformation]
- $\mathbf{NP-complete}$: 1) $A \in \mathbf{NP}$ & 2) $\forall B \in \mathbf{NP}, B \leq_{Karp,P} A$ [$\mathbf{NP}-\mathrm{hard}$]
  - $\mathrm{Circuit-SAT}$ is $\mathbf{NPC}$
  - $\mathrm{Circuit-SAT} \leq \mathrm{3-SAT} \leq \mathrm{INDEPENDENT-SET} \leq \mathrm{VERTEX-COVER} \leq \mathrm{SET-COVER} \leq \mathrm{IntegerLP}$
  - $\mathrm{3-SAT} \leq \mathrm{HAM-CYCLE}$

Coping with NPC: approximation algorithms

- Greedy
  - Vertex cover & set cover

- LP relaxation
  - Threshold rounding: 2-approx. vertex cover
  - Randomized rounding: $O(\log n)$-approx. set cover

★ Know the facts and ideas! Details less important

FAQs

- How should I study for it?
  - Review the fundamentals
  - Reproduce the algorithms & analysis for all problems you've seen (lecs, text, hw…)
- Reminders
  - If no running time requirement, always aim for fastest algorithms you can think of.
  - Asked or not, always provided analysis (correctness and runtime) on algorithm design problems.
  - Always start with a short description of the main idea of your algorithm.
  - Reductions: mind the direction (e.g., in NPC proofs).
  - A guideline on grading rubrics will be posted.
- Questions?