CS 410/510 Introduction to Quantum Computing

Homework 6

Portland State U, Spring 2020 — *05/17/2020*

Student: your name — *Due: 11:59pm PDT, **05/31/2020***

Instructions. This problem set contains 17 pages (including this cover page) and 4 questions. Problems marked with "[**G**]" are required for 510 students. Students enrolled in 410 will get bonus points for solving them. A random subset of problems will be graded.

Your solutions will be graded on *correctness* and *clarity*. You should only submit work that you believe to be correct, and you will get significantly more partial credit if you clearly identify the gap(s) in your solution. It is good practice to start any long solution with an informal (but accurate) summary that describes the main idea.

You need to submit a PDF file via Gradescope before the deadline. Either a clear scan of you handwriting or a typeset document is accepted. You will get 5 bonus points for typing in LaTeX (Download and use the accompany TeX file).

You may collaborate with others on this problem set. However, you must ***write up your own solutions*** and ***list your collaborators*** for each problem.

1. (Search for a quantum state) Suppose you are given a black box U_ϕ that identifies an unknown quantum state $|\phi\rangle$ (which may not be a computational basis state). Specifically, $U_\phi|\phi\rangle = -|\phi\rangle$, and $U_\phi|\xi\rangle = |\xi\rangle$ for any state $|\xi\rangle$ satisfying $\langle\phi|\xi\rangle = 0$. Consider an algorithm for preparing $|\phi\rangle$ that starts from some fixed state $|\psi\rangle$ and repeatedly applies the unitary transformation VU_ϕ, where $V = 2|\psi\rangle\langle\psi| - I$ is a reflection about $|\psi\rangle$. Let

$$|\phi^\perp\rangle = \frac{e^{-i\lambda}|\psi\rangle - \sin(\theta)|\phi\rangle}{\cos(\theta)}$$

denote a state orthogonal to $|\phi\rangle$ in $span\{|\phi\rangle, |\psi\rangle\}$, where $\langle\phi|\psi\rangle = e^{i\lambda}\sin(\theta)$ for some $\lambda, \theta \in \mathbb{R}$.

   (a) (4 points) Write the initial state $|\psi\rangle$ in the basis $\{|\phi\rangle, |\phi^\perp\rangle\}$.

(b) (6 points) Write U_ϕ and V as matrices in the basis $\{|\phi\rangle, |\phi^\perp\rangle\}$.

(c) (5 points) Let k be a positive integer. Compute $(VU_\phi)^k$.

(d) (5 points) Compute $\langle \phi | (VU_\phi)^k | \psi \rangle$.

(e) (5 points) Suppose that $|\langle\phi|\psi\rangle|$ is small. Approximately what value of k should you choose in order for the algorithm to prepare a state close to $|\phi\rangle$, up to a global phase? Express your answer in terms of $|\langle\phi|\psi\rangle|$.

2. (Mixed states and density matrix)

   (a) (5 points) A density matrix ρ corresponds to a pure state if and only if $\rho = |\psi\rangle\langle\psi|$ for some $|\psi\rangle$. Show that ρ corresponds to a pure state if and only if $Tr(\rho^2) = 1$.

(b) (5 points) Show that every 2×2 density matrix ρ can be expressed as an equally weighted mixture of pure states. That is $\rho = \frac{1}{2}|\psi_1\rangle\langle\psi_1| + \frac{1}{2}|\psi_2\rangle\langle\psi_2|$ (Note: the two states need not be orthogonal).

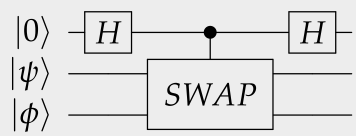
(c) (5 points) Imagine two parties Alice and Bob. Alice flips a biased coin which is HEADS with probability $\cos^2(\pi/8)$. Alice prepares $|0\rangle$ when she sees coin 0 and $|1\rangle$ otherwise. From Alice's perspective (who knows the coin value), the density matrix of the state she created will be either $|0\rangle\langle 0|$ or $|1\rangle\langle 1|$. She then sends the qubit to Bob. What is the density matrix of the state from Bob's perspective (who does not know the coin value)? Write down the matrix.

(d) (5 points (bonus)) Let $\{p_i, |\psi_i\rangle\}_{i=0}^{1}$ and $\{q_j, |\phi_j\rangle\}_{j=0}^{1}$ be two ensembles of pure states. Define $|\tilde{\psi}_i\rangle = \sqrt{p_i}|\psi_i\rangle$ and $|\tilde{\phi}_j\rangle = \sqrt{q_j}|\phi_j\rangle$ for all i, j. Show that the two ensembles produce the same density matrix **if and only if** there is a unitary $U = \begin{pmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{pmatrix}$ such that

$$|\tilde{\psi}_0\rangle = u_{00}|\tilde{\phi}_0\rangle + u_{01}|\tilde{\phi}_1\rangle, \text{ and } |\tilde{\psi}_1\rangle = u_{10}|\tilde{\phi}_0\rangle + u_{11}|\tilde{\phi}_1\rangle\,.$$

3. (Swap test) Let $|\psi\rangle$ and $|\phi\rangle$ be arbitrary single-qubit states (not necessarily computational basis states), and let SWAP denote the 2-qubit gate that swaps its input qubits (i.e., $\text{SWAP}|x\rangle|y\rangle = |y\rangle|x\rangle$ for any $x, y \in \{0, 1\}$).

   (a) (5 points) Compute the output state of the following quantum circuit:



(b) (5 points) Suppose the top qubit in the above circuit is measured in the computational basis. What is the probability that the measurement result is 0 and what is the posterior state of the remaining two qubits in this case?

(c) (5 points) Suppose the two input qubits are in a mixed state $\rho \otimes \sigma$. What is the probability of measuring 0 if we run the circuit on them?

(d) (5 points (bonus)) How do the results of the previous parts change if the bottom two registers are n-qubit states, and swap denotes the $2n$-qubit gate that swaps the first n qubits with the last n qubits?

4. (5-qubit code) Consider a quantum error correcting code that encodes one logical qubit into five physical qubits, with the logical basis states

$$\begin{aligned}
|0_L\rangle =& \frac{1}{4}(|00000\rangle \\
&+|10010\rangle + |01001\rangle + |10100\rangle + |01010\rangle + |00101\rangle \\
&-|11000\rangle - |01100\rangle - |00110\rangle - |00011\rangle - |10001\rangle \\
&-|01111\rangle - |10111\rangle - |11011\rangle - |11101\rangle - |11110\rangle) \\
|1_L\rangle =& \frac{1}{4}(|11111\rangle \\
&+|01101\rangle + |10110\rangle + |01011\rangle + |10101\rangle + |11010\rangle \\
&-|00111\rangle - |10011\rangle - |11001\rangle - |11100\rangle - |01110\rangle \\
&-|10000\rangle - |01000\rangle - |00100\rangle - |00010\rangle - |00001\rangle)
\end{aligned}$$

(a) (5 points) Show that $|0_L\rangle$ and $|1_L\rangle$ are simultaneous eigenstates (with eigenvalue +1) of the operators below. (Hint: You can show this without explicitly checking every case.)

$$\begin{aligned}
&X \otimes Z \otimes Z \otimes X \otimes I \\
&I \otimes X \otimes Z \otimes Z \otimes X \\
&X \otimes I \otimes X \otimes Z \otimes Z \\
&Z \otimes X \otimes I \otimes X \otimes Z \\
&Z \otimes Z \otimes X \otimes I \otimes X
\end{aligned}$$

(b) (5 points) Show that this code can correct an X or Z error acting on any of the five qubits. You should explain how the different possible errors would be reflected by a measurement of the error syndrome.

(c) (5 points) Find logical Pauli operators X_L and Z_L such that

$$X_L|0_L\rangle = |1_L\rangle, X_L|1_L\rangle = |0_L\rangle; \quad Z_L|0_L\rangle = |0_L\rangle, Z_L|1_L\rangle = -|1_L\rangle\,.$$