CS 410/510 Introduction to Quantum Computing

Homework 1

Portland State U, Spring 2020	*04/05/2020*
Lecturer: Fang Song	*Due: 11:59pm PDT, 04/12/2020*

Instructions. This problem set contains 8 pages (including this cover page) and 2 questions. A random subset of problems will be graded.

Your solutions will be graded on *correctness* and *clarity*. You should only submit work that you believe to be correct, and you will get significantly more partial credit if you clearly identify the gap(s) in your solution. It is good practice to start any long solution with an informal (but accurate) summary that describes the main idea.

You need to submit a PDF file via Gradescope before the deadline. Either a clear scan of you handwriting or a typeset document is accepted. You will get 5 bonus points for typing in LaTeX (Download and use the accompany TeX file).

You may collaborate with others on this problem set. However, you must ***write up your own solutions*** and ***list your collaborators*** for each problem.

1. (Basic algebra)

   (a) (6 points) (complex number) For complex number $c = a + bi$, recall that the real and imaginary parts of c are denoted $Re(c) = a$ and $Im(c) = b$.

      i) Prove that $c + c^* = 2 \cdot Re(c)$.

      ii) Prove that $|c|^2 := cc^* = a^2 + b^2$.

      iii) What is the polar form of $c = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$? Use the fact that $e^{i\theta} = \cos\theta + i\sin\theta$.

(b) (6 points) (Trace) Recall the trace of a matrix $M = (m_{ij})_{n\times n}, m_{ij} \in \mathbb{C}$ is defined by $\mathrm{tr}(M) := \sum_{i=1}^{n} m_{ii}$. Let $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $Y = iXZ$.

i) Show that $\mathrm{tr}(YZ) = \mathrm{tr}(ZY)$.

ii) Prove that this holds for general matrices: any $n \times n$ matrices M and N, $\mathrm{tr}(MN) = \mathrm{tr}(NM)$.

(c) (6 points) (Inner/outer product)

i) Let $|\phi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$. $|\psi\rangle = \sqrt{\frac{2}{5}}|0\rangle - \sqrt{\frac{3}{5}}|1\rangle$. Calculate $\langle\phi|\psi\rangle$.

ii) Show that $X = |0\rangle\langle 1| + |1\rangle\langle 0|$. Express Y, Z in this outer product form too.

(d) (6 points) (Basis and eigenvectors)

i) Show that $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ are both eigenvectors of X. What are their respective eigenvalues?

ii) Show that $\{|+\rangle, |-\rangle\}$ form an orthonormal basis of $\mathbb{C}^2$.

(e) (10 points (bonus)) Recall that a matrix H is called Hermitian if $H = H^\dagger$ (where $\dagger$ denotes the conjugate transpose), and a matrix U is called unitary if $U^\dagger = U^{-1}$ (the matrix inverse of U). The matrix exponential is defined by its Taylor series as $\exp(A) = \sum_{j=0}^{\infty} \frac{A^j}{j!}$. Prove that if H is Hermitian, then $\exp(iH)$ is unitary.

2. (Qubit) Let $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

(a) (12 points) Consider the following operations on a qubit.

i) Apply H to the qubit $\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$. What is the resulting state?

ii) Next continue to apply Z to this qubit. What is the resulting state?

iii) Finally make a measurement (in the computational basis), describe the full effect of the measurement.

(b) (4 points) Suppose we have a qubit and we first apply X and then Z. Is it equivalent to first applying Z and then X? Justify your answer.