Portland State University

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S'20 CS410/510
Intro to quantum computing

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Week 9

- Quantum error correction
- Quantum fault-tolerance

Credit: based on slides by Richard Cleve

Recall: quantum channels

$$\rho \;\text{———}\; \boxed{\Phi} \;\text{———}\; \rho'$$

Let $A_1, A_2, \ldots, A_m$ be matrices satisfying $\sum_{j=1}^{m} A_j^\dagger A_j = I$.

Then the mapping $\rho \mapsto \sum_{j=1}^{m} A_j \rho A_j^\dagger$ is a general quantum operator.

- N.B. A_i need NOT be square matrices
- Also known as quantum channels

Examples of quantum channels

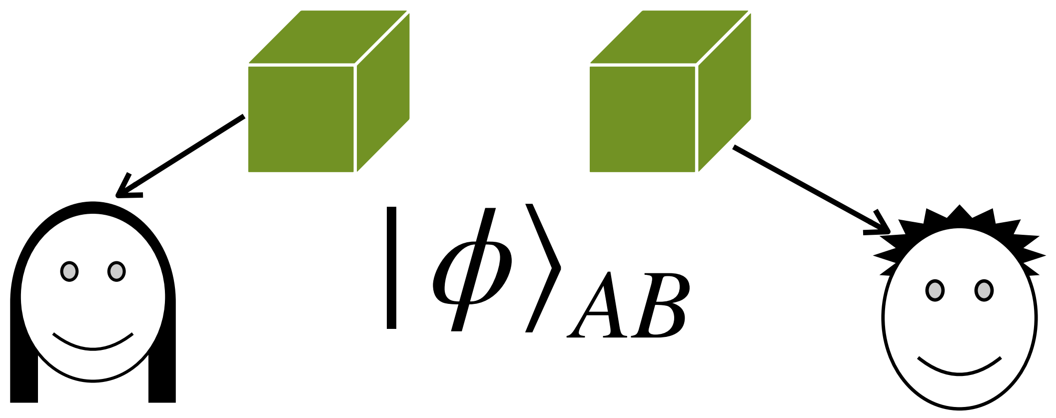
3. **Partial trace** $A_0 = I \otimes \langle 0 | = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, A_1 = I \otimes \langle 1 | = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

- Check validity:

- Apply to $|0\rangle\langle 0| \otimes | + \rangle\langle + |$

- Apply to $|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

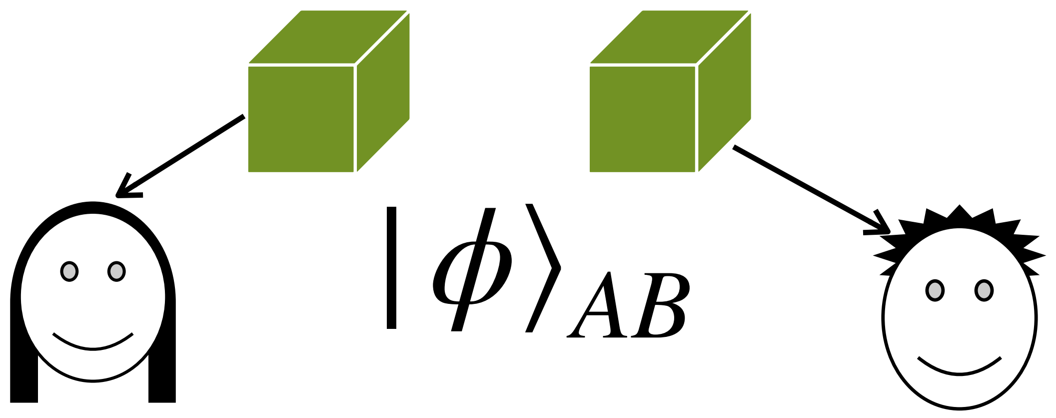
Exercise



1. let Tr_B denote partial trace of subsystem B. Suppose Alice and Bob shares two qubits in state $|\phi\rangle_{AB}$.

- Apply Tr_B to $|\phi\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
- Apply Tr_B to $|\phi\rangle_{AB} = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$
- Is Alice able to tell the two cases on her side?

Exercise



2. let Tr_B denote partial trace of subsystem B. Suppose Alice and Bob shares two qubits in state $|\phi\rangle_{AB}$.

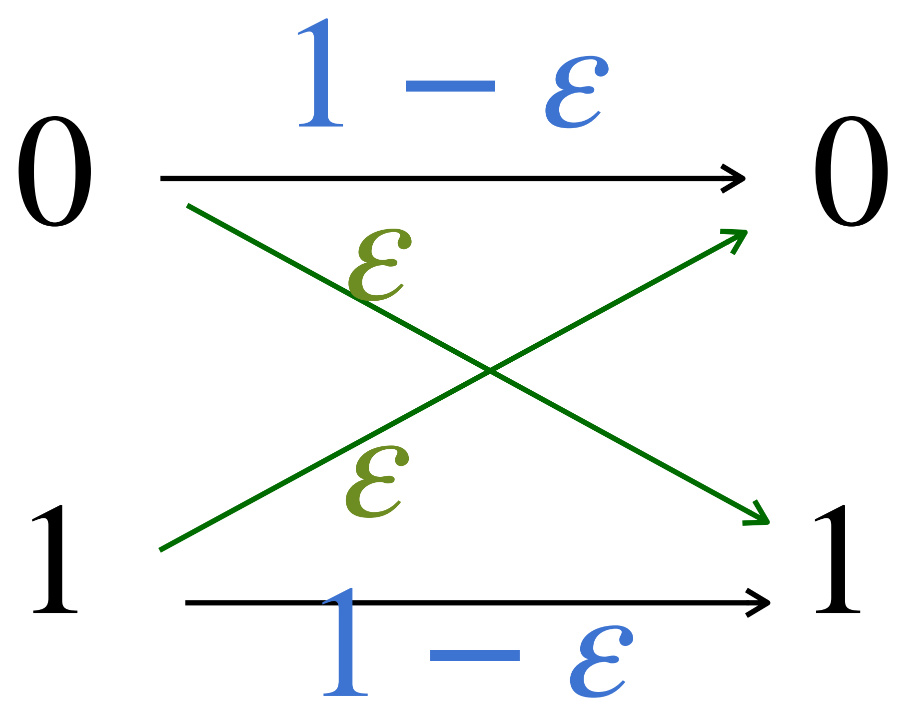
- Apply Tr_B to $|\phi\rangle_{AB} = \frac{3}{5}|00\rangle + \frac{4}{5}|11\rangle$
- Apply Tr_B to $|\phi\rangle_{AB} = \frac{4}{5}|00\rangle - \frac{3}{5}|11\rangle$
- Is Alice able to tell the two cases on her side?

Error correction codes

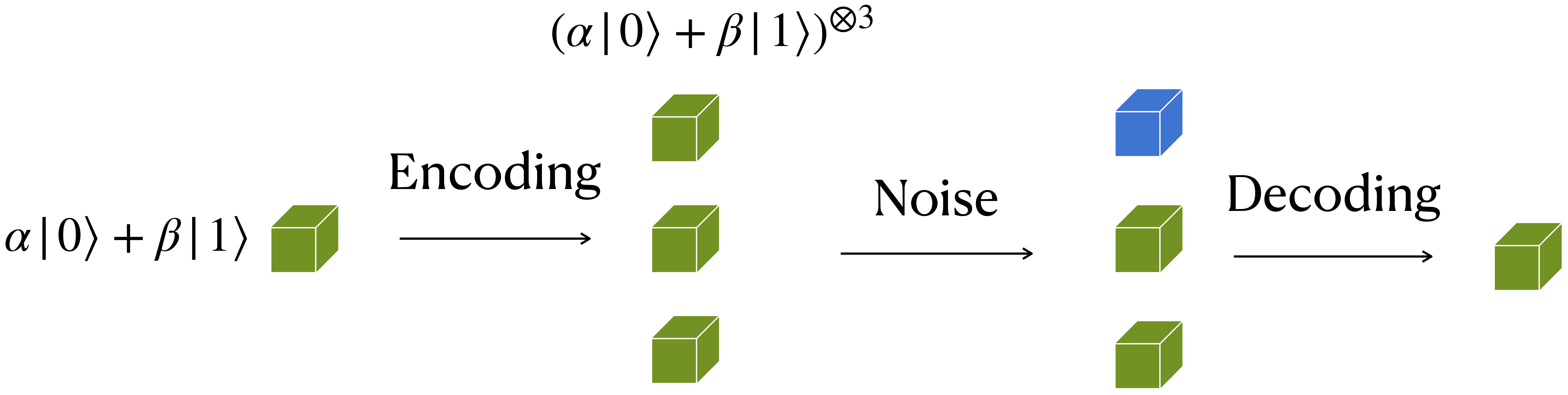
Classical error correcting codes (ECC)

- **Protecting data against noises during transmitting or storing**

- **Binary symmetric channel: each bit flips w. probability ε independently**
  - A simple noise model, reality may be more complex and unpredictable



Quantum repetition code?



:(This would violate no-cloning …

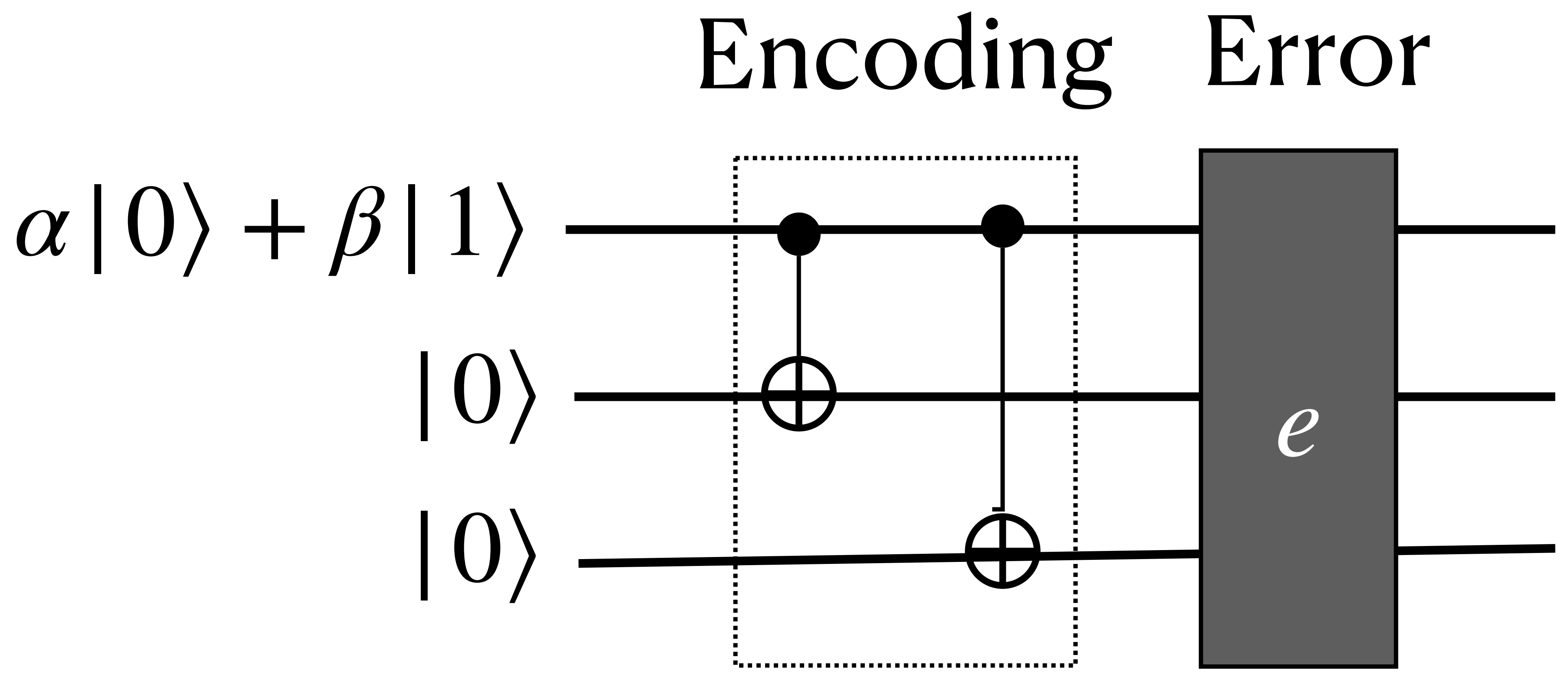
3-bit repetition code

- **Redundancy is our friend**
  - $E : b \mapsto bbb$; repete to encode
  - $D : b_1 b_2 b_3 \mapsto maj(b_1, b_2, b_3)$; take majority to decode
- **Effective error probability reduces from ε to $3\varepsilon^2 - 2\varepsilon^3$**

ε	$3\varepsilon^2 - 2\varepsilon^3$	**Error reduced by a factor of**
0.1	0.009	11
0.01	0.0001	100
0.001	0.0000001	1000

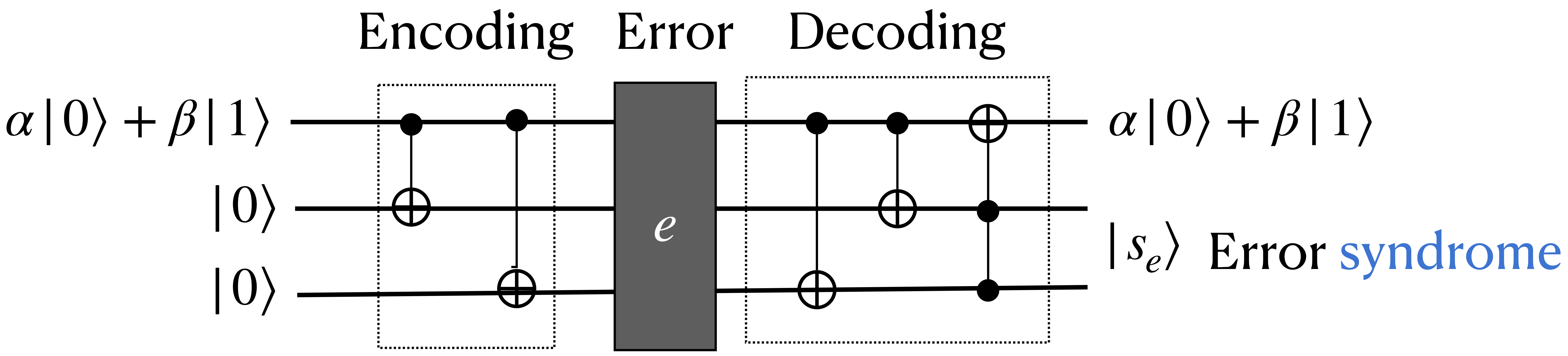
3-qubit code for one X-error

- **Encoding E**
  - $|0\rangle \mapsto |0_L\rangle := |000\rangle, |1\rangle \mapsto |1_L\rangle := |111\rangle$
  - $\alpha|0\rangle + \beta|1\rangle \mapsto \alpha|000\rangle + \beta|111\rangle$



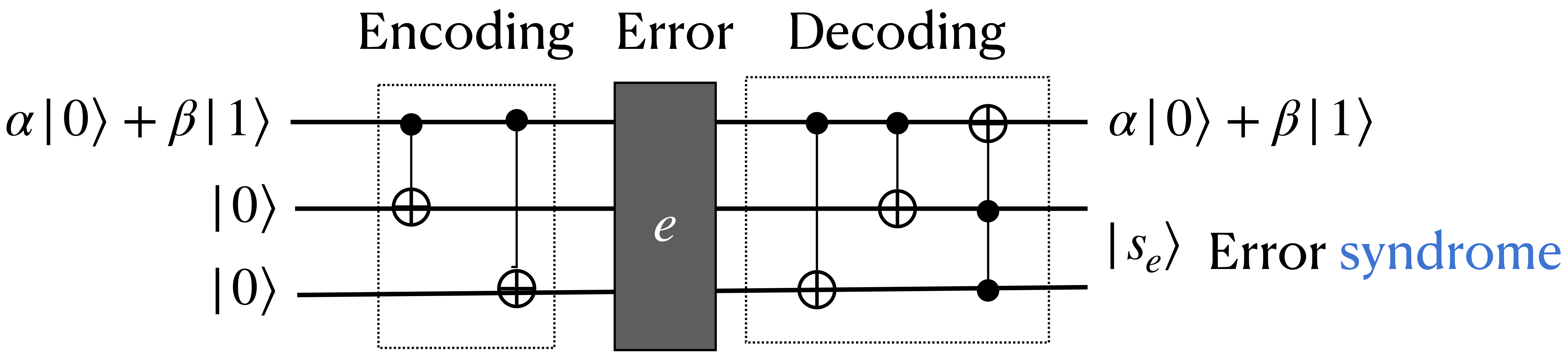
- **What if a quantum bit-flip error?**
  - $I \otimes I \otimes I \quad X \otimes I \otimes I \quad I \otimes X \otimes I \quad I \otimes I \otimes X$

3-qubit code for one X-error



Error	$I \otimes I \otimes I$	$X \otimes I \otimes I$	$I \otimes X \otimes I$	$I \otimes I \otimes X$
Error syndrome	$|00\rangle$	$|11\rangle$	$|10\rangle$	$|01\rangle$

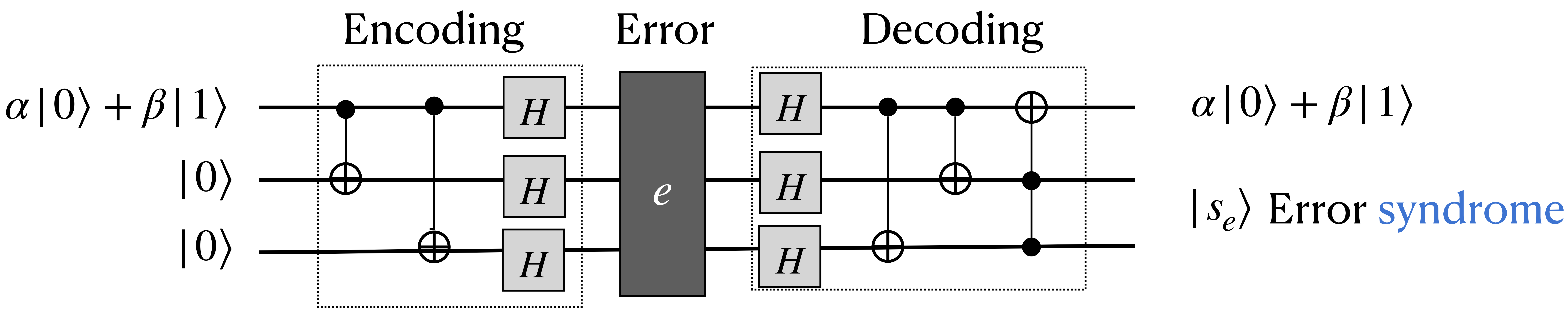
Does it help with Z-error?



- **Example.** $e = Z \otimes I \otimes I$

3-qubit code for one Z-error

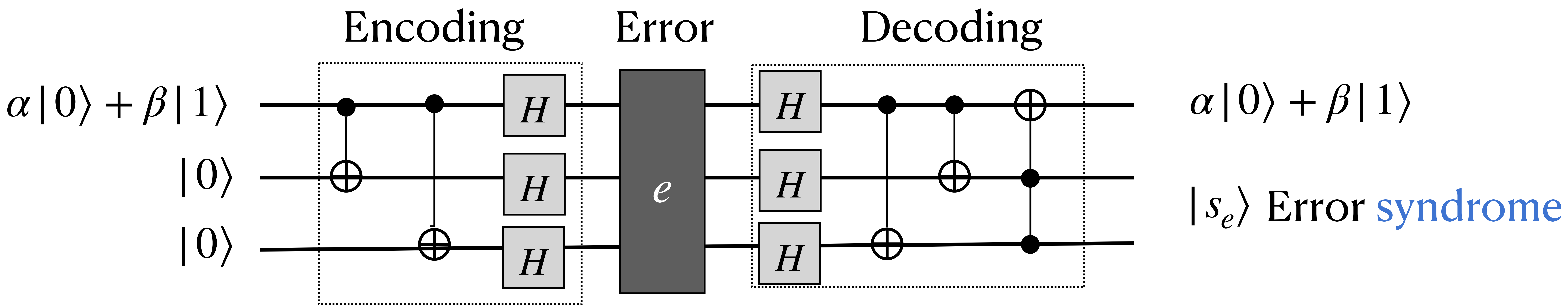
- **Observation.** $HZH = X$. Reducing Z-erro to X-error



- **Encoding** E. $|0\rangle \mapsto |0_L\rangle := |+++\rangle, |1\rangle \mapsto |1_L\rangle := |---\rangle$

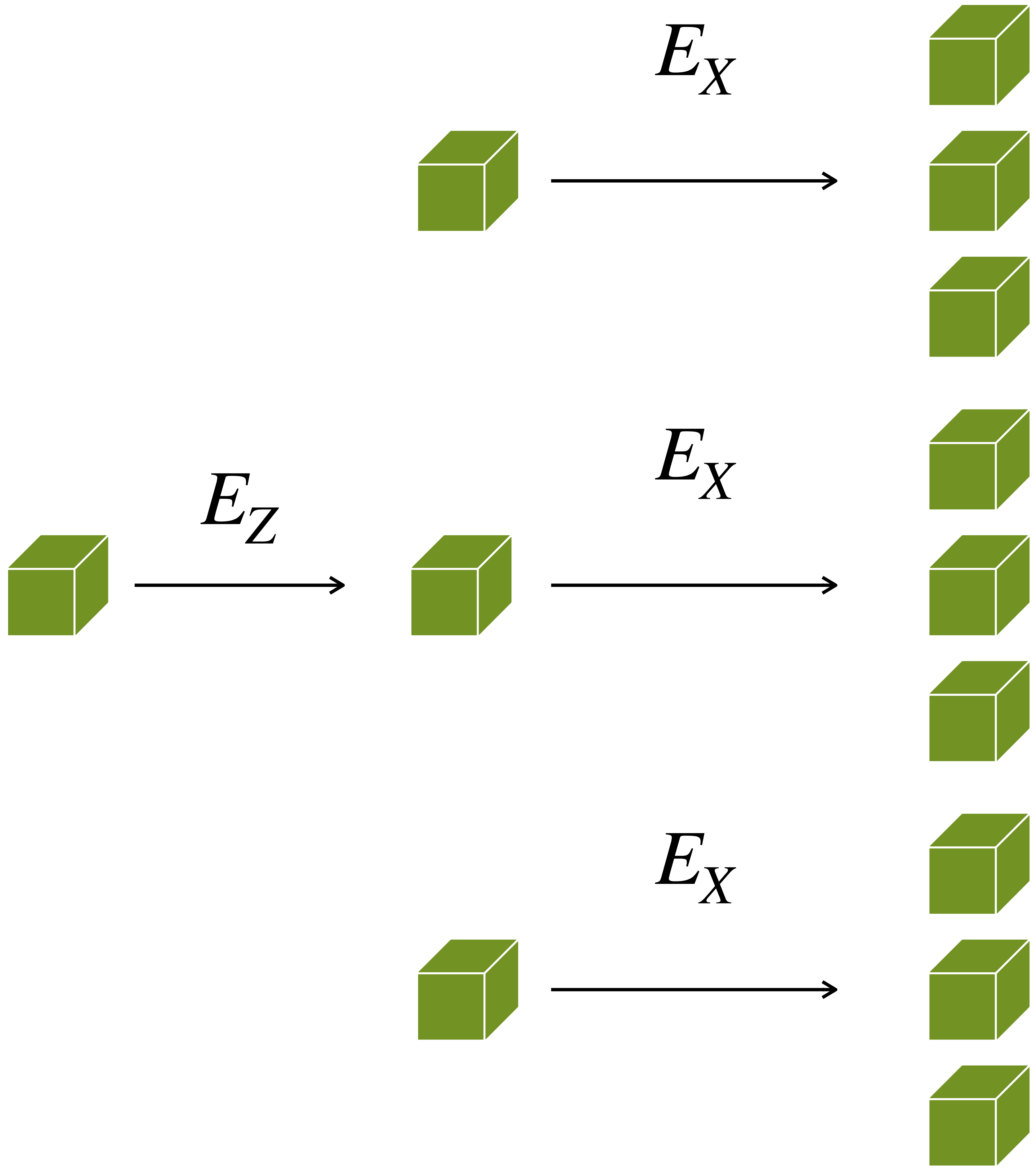
Error	$I \otimes I \otimes I$	$X \otimes I \otimes I$	$I \otimes X \otimes I$	$I \otimes I \otimes X$
Error syndrome	$\|00\rangle$	$\|11\rangle$	$\|10\rangle$	$\|01\rangle$

Does it help with X-error?



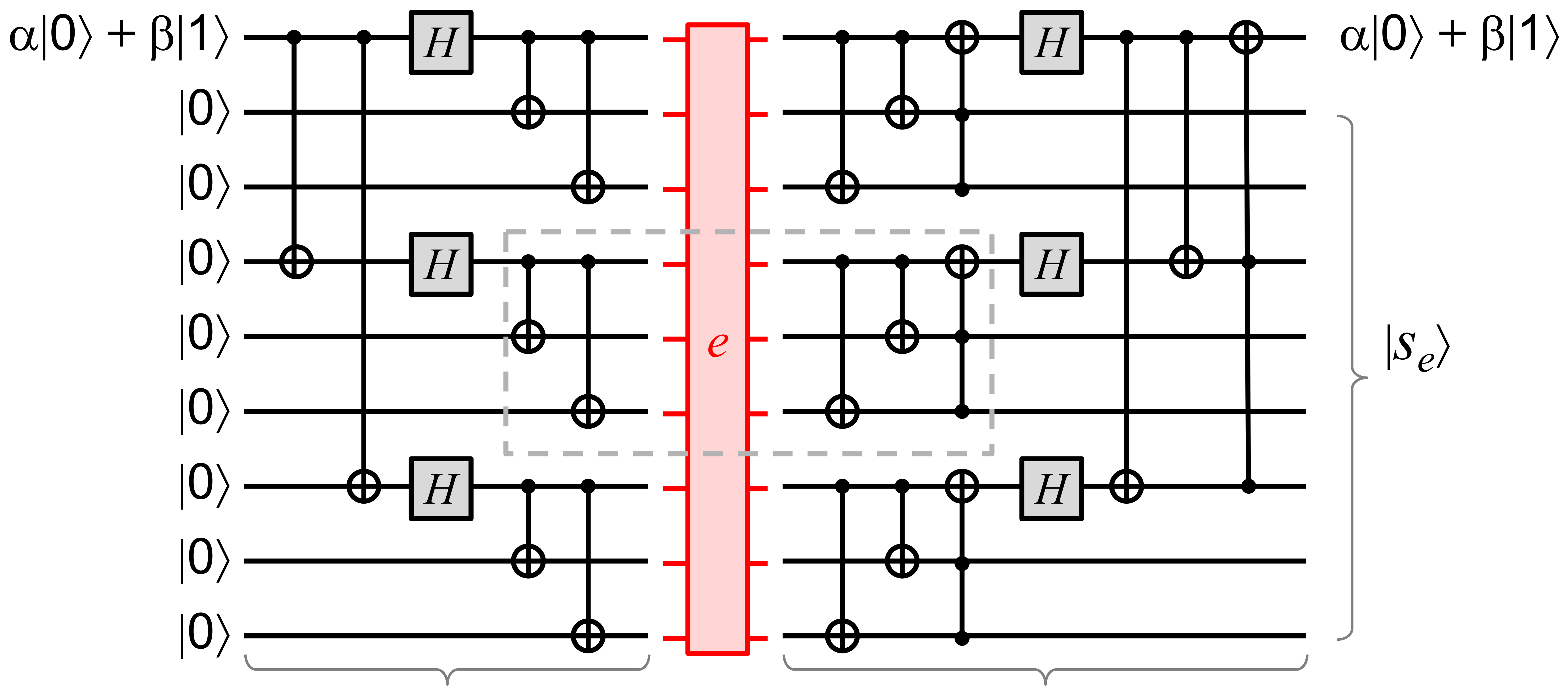
- **Example.** $e = X \otimes I \otimes I$

Shor's 9-qubit code



- $|0\rangle \mapsto |+++\rangle \mapsto \left(\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)\right)^{\otimes 3} =: |0_L\rangle$
- $|1\rangle \mapsto |---\rangle \mapsto \left(\frac{1}{\sqrt{2}}(|000\rangle - |111\rangle)\right)^{\otimes 3} =: |1_L\rangle$

Shor's 9-qubit code



- Able to correct a single X or Z error
  - "Inner " part corrects any single-quit X error
  - "Inner " part corrects any single-quit X error
- Since $Y = iXZ$, single-quit Y-error can be corrected too

Arbitrary one-qubit errors

- **Observation. Any one-qubit unitary U can be written as $U = \lambda_0 I + \lambda_1 X + \lambda_2 Y + \lambda_3 Z$ for some $\lambda_i \in \mathbb{C}$.**

$$\alpha|0\rangle + \beta|1\rangle \overset{E}{\mapsto} \alpha|0_L\rangle + \beta|1\rangle_L \overset{I\otimes U\otimes\ldots\otimes I}{\mapsto} |\tilde{\psi}\rangle$$

$$\overset{D}{\mapsto} (\alpha|0\rangle + \beta|1\rangle)(\lambda_0|s_I\rangle + \lambda_1|s_X\rangle + \lambda_2|S_Y\rangle + \lambda_3|S_Z\rangle)$$

- **Corollary. Shor's 9-qubit code protects against any one-qubit unitary error. In fact the error can be any one-qubit quantum channel Φ.**

- **More QECC: CSS codes & stabilizer codes**
  - 5-qubit code: optimal for correcting single-qubit errors
  - Surface code: elegant theory and promising in realization

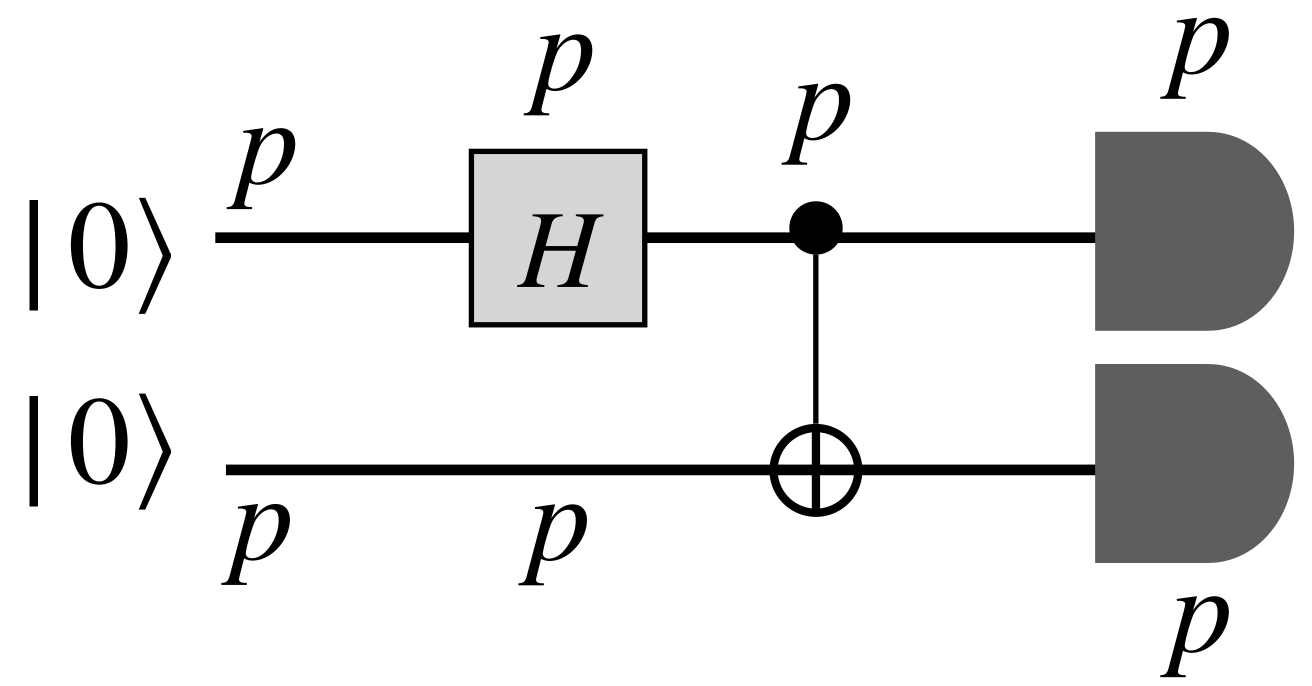
Fault-tolerant computing

Error is ubiquitous

QECC solves the problem of storing and transmitting quantum information.

But we want to do more: computation on them

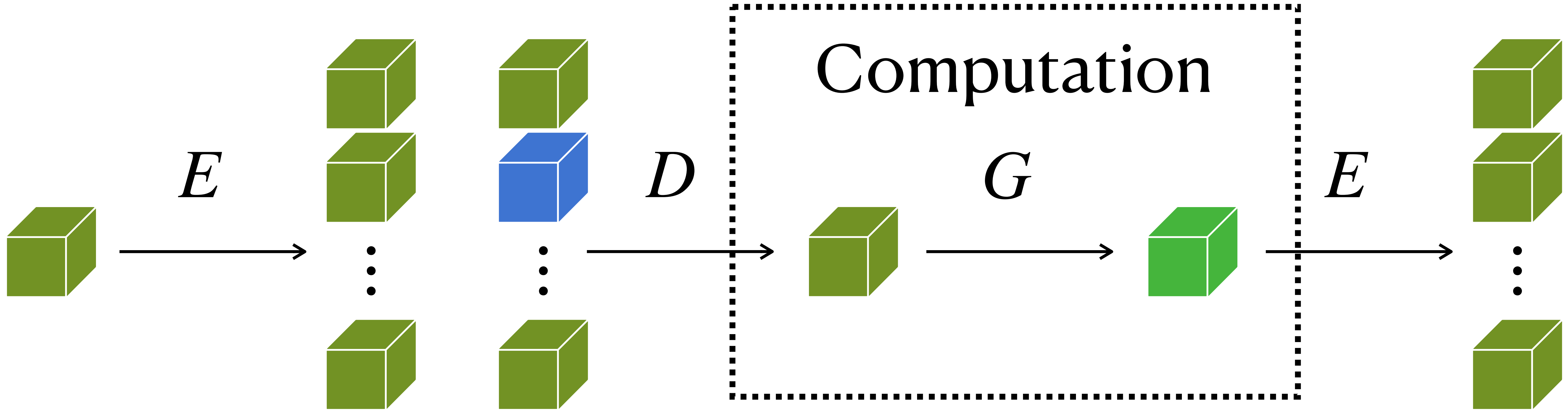
- Observation. Any "location" can "fail".
  - Gate, measurement, storage, prep, ...



- Simple error model: each location fails with probability p
  - Circuit of size ℓ. $\Pr[\text{no error}] =$

Attempt 1

- Enc — Dec — Compute — Enc

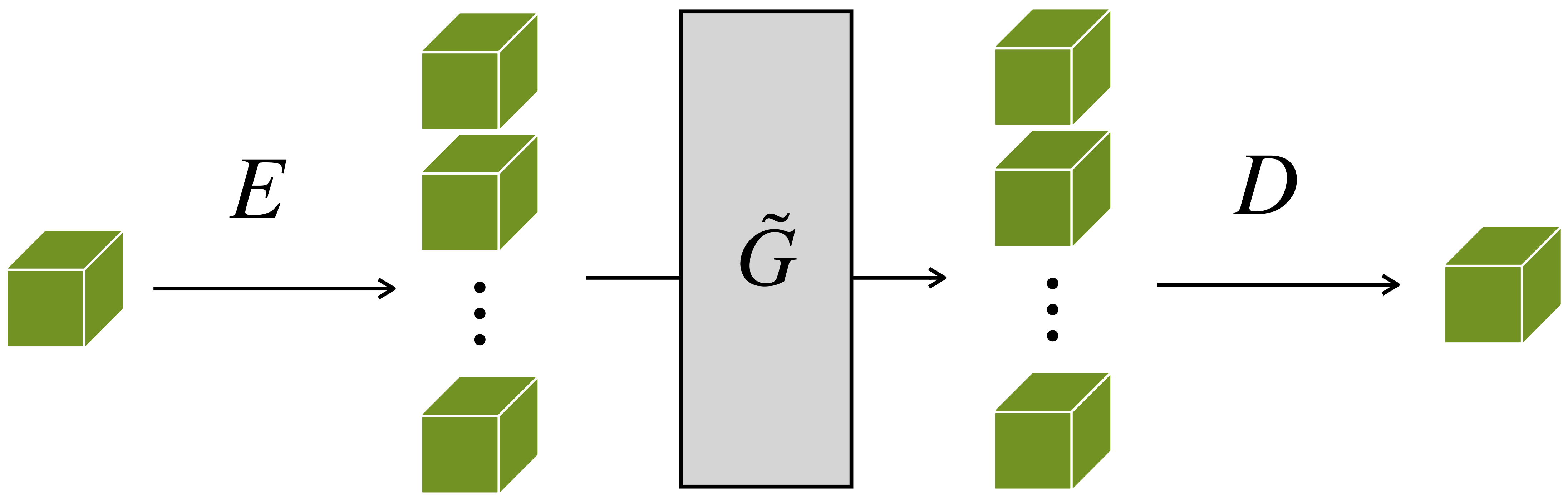


- Drawback:

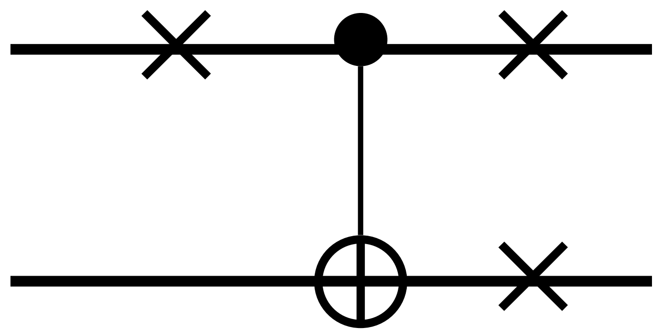
Attempt 2

- **Computing on encoded data**

Encoded gate

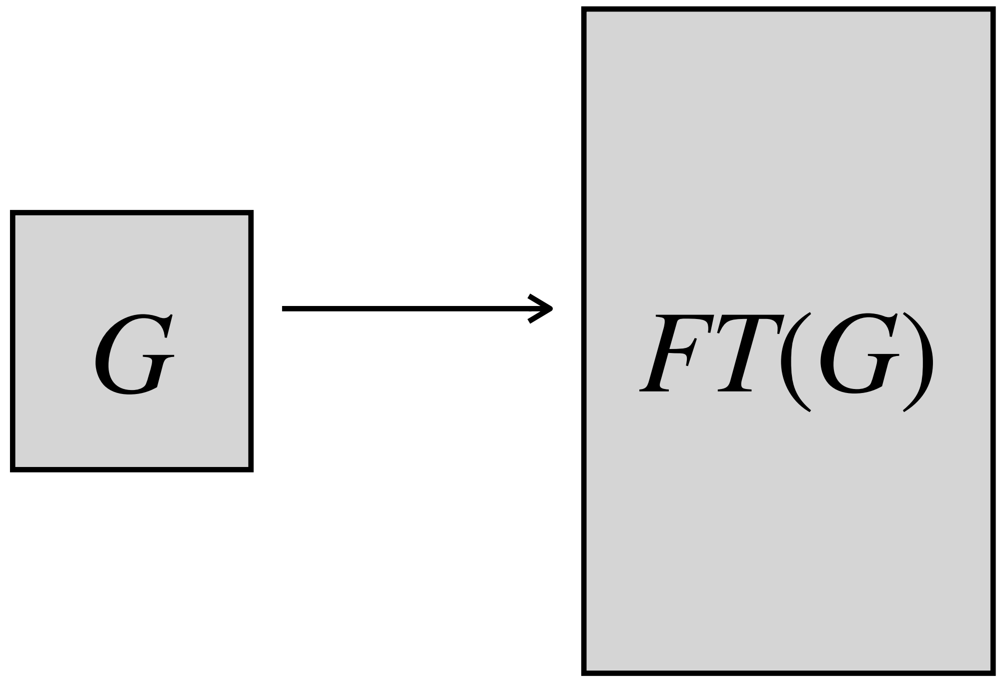


- **Challenges**
  - Non-perfect $\tilde{G}$: ok if not many
  - Error propagation



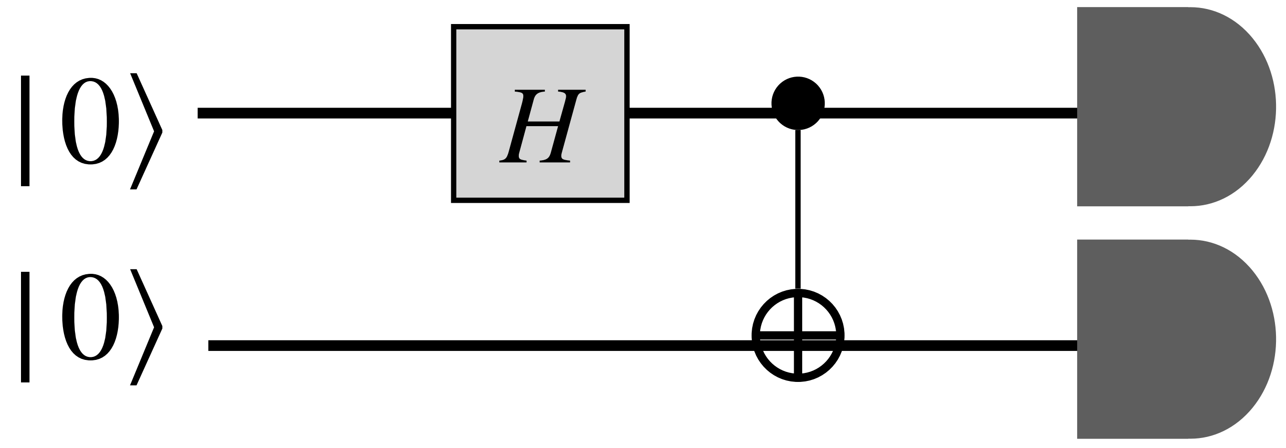
Fault-tolerant gadget

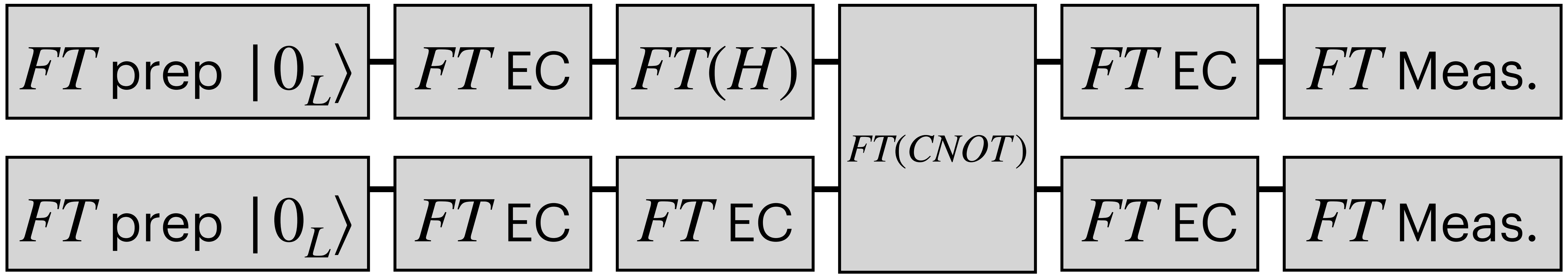
- When designed encoded gates, make sure not to introduce too many errors
  - FT gate,
  - FT state prep
  - FT measurement



Specific to the error-correcting code used

- Putting it together: FT operations + Frequent FT error-correcting





Threshold theorem

Theorem. There is a fixed constant p_{th} such that a circuit of size T can be translated to a circuit of size $O(T\log T)$ that is robust against the error model with error $p \le p_{th}$.

- p_{th} depends heavily on the QECC
  - Steane code: $\sim 10^{-5}$
  - Surface code: $\sim 10^{-2}$
- Another key idea: concatenation

Quantum computational complexity

Encounters so far

- Computability: can you solve it, in principle?

[Given program code, will this program terminate or loop indefinitely?]

Uncomputable!

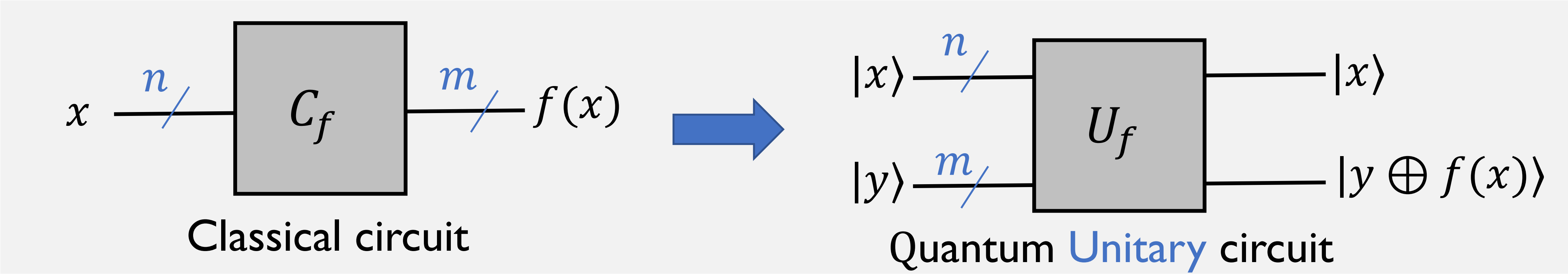
Church-Turing Thesis. A problem can be computed in any ***reasonable*** model of computation iff. it is computable by a **Boolean circuit**.

- Complexity: can you solve it, under resource constraints?

[Can you factor a 1024-bit integer in 3 seconds?]

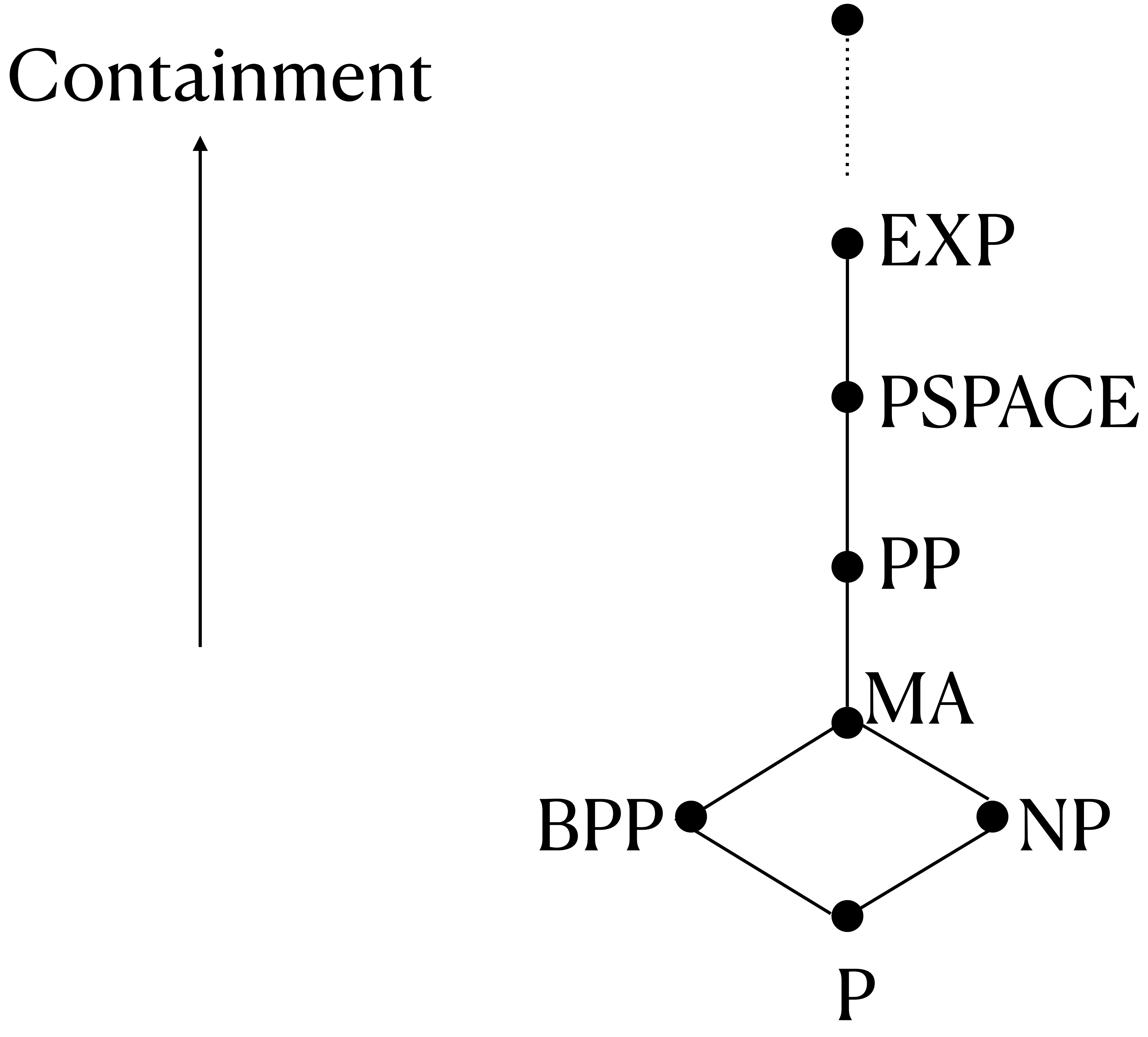
Extended Church-Turing Thesis. A function can be computed efficiently in any *reasonable* model of computation iff. it is efficiently computable by a **Boolean circuit**.

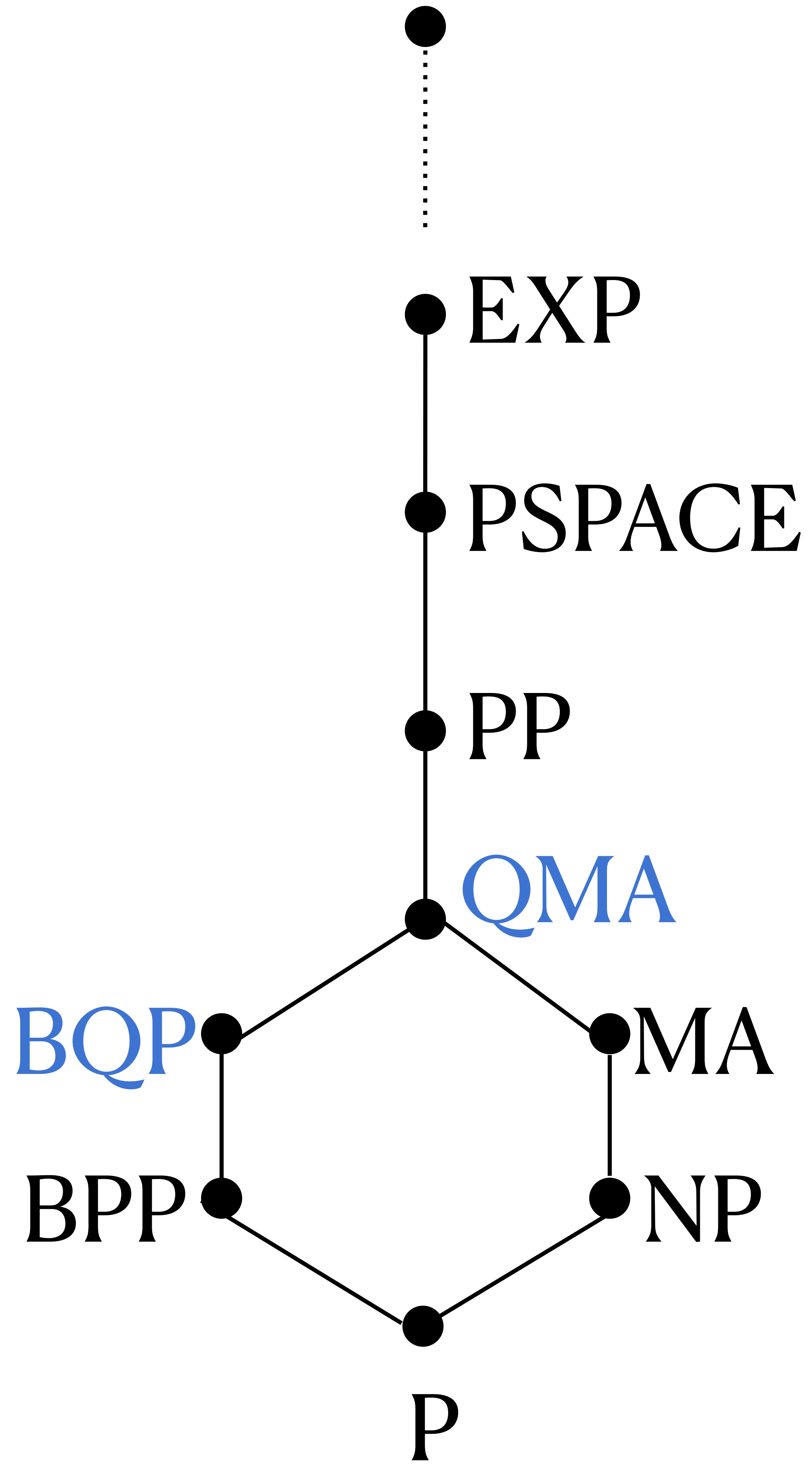
✓ Quantum computer Disprove ECTT?



Corollary. BPP⊆BQP [More to come in future]

Landscape of complexity classes





Discussion: quantum party is on!?

- What do you think about its description of quantum computing?

- Think of a few local companies. Can you identify where quantum computers might help them?

Looking forward to your presentations!

Scratch