Portland State U

F, 05/01/2020

S′20 CS 410/510

Intro to quantum computing

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Week 5

- Modular arithmetic
- Order finding
- Prime factorization

Credit: based on slides by Richard Cleve

Exercise

1. Compute the product of the numbers below
   - Example. $3 \times 5 = 15$
   - $19 \times 31 =$ 589
   - $244176193 \times 176944583 =$ . . . .
2. Find the prime factorization of the numbers below.
   - Example. $15 = 3 \times 5$
   - $21 =$ 3x7
   - $247 =$ 13x19
   - $205027 =$ 421 x 487
   - $55514685797288803 =$ . . . . .
3. How many bits do we need to write an integer $x \in \mathbb{Z}$ in binary?

$\log_2 x$ $\log x$

A round of applause

- Exponential quantum speedup
  - Nice, but query-model, "artificial" problems …

Black-box problem	Deterministic	Randomized	Quantum
Deutsch	2 (queries)	2 (queries)	1 (query)
Deutsch-Josza	$2^{n-1}+1$	$\Omega(n)$	1 (Exact)
Simon	$2^{n-1}+1$	$\Omega(\sqrt{2^n})$	$O(n)$

- Today: quantum (exponential) speedup on a "real-life" hard problem

Integer factorization

Input. Positive integer $N (= pq, p, q\ prime)$

Goal. Find p, q

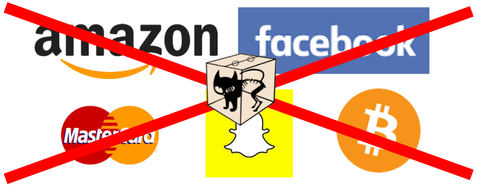
- Classical efficient algorithm NOT known
  - Number field sieve ~ $2^{O(\underline{(\log N)}^{\frac{1}{3}}(\log\log N)^{\frac{2}{3}})}$

Efficient = poly-time in input size
Ex. N has n bits. Runtime $O(n^5)$

- Efficient quantum algorithm $O((\log N)^3)$ [Shor94, Kitaev94]
  - Generalization of Simon's algorithm

An inconvenient consequence in cybersecurity

- RSA cryptosystem relies on hardness of factorization
  - Foundation of modern cryptography and Internet security



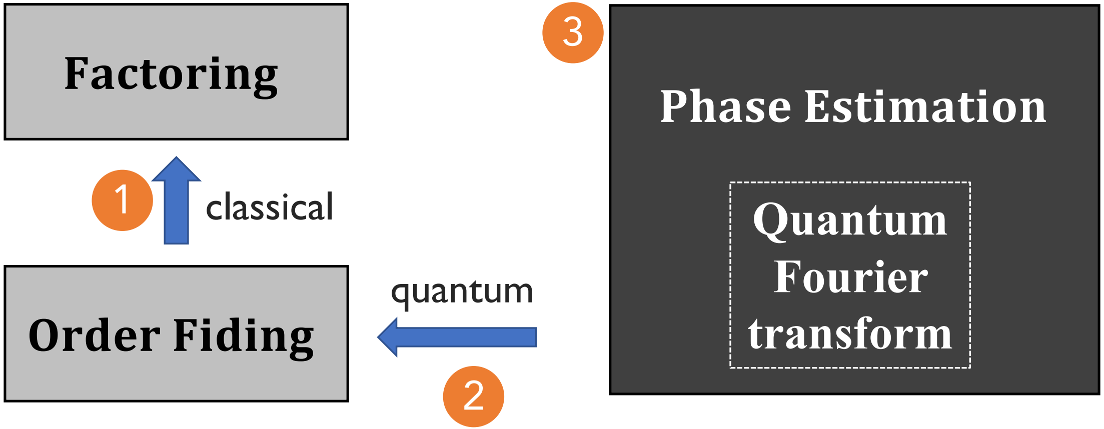
Will be broken by a quantum computer

- RSA Factoring Challenge

Feb 28, 2020: RSA-250 (250 decimal digits = 829 bits) factored!
Total computation time ~ 2700 core-years (Intel Xeon Gold 6130)

RSA-250 = 6413528947707158027879019017057738908482501474294344720811685963202453234463
0238623598752668347708737661925585694639798853367 ×
3337202759497815655622601060535511422794076034476755466678452098702384172921
0037080257448673296881877565718986258036932062711

Roadmap to quantum factorization algorithm



- Today: 1 & 2 (treating PE as black-box)
- Next time: 3 open up PE and QFT

Review: arithmetic/number theory

Modular arithmetic

$$a, b, N \in \mathbb{Z}, N \geq 1$$

- $a \equiv b \bmod N \Leftrightarrow N \mid (a - b)$
- $\gcd(a, b) = \max\{c: c|a \text{ and } c|b\}$
  - a, b coprime, if $\gcd(a, b) = 1$
- $\mathbb{Z}_N := \{0, 1, \dots, N - 1\}$
- $\mathbb{Z}_N^* := \{a \in \mathbb{Z}_N : \gcd(a, N) = 1\}$
  - Euler φ function $\varphi(N) := |\mathbb{Z}_N^*|$
- Fact. $\forall\, a \in \mathbb{Z}_N^*, \exists!\, (\text{unique}) b \in \mathbb{Z}_N^*\ s.t.\ ab \equiv 1 \bmod N$
  - Call b the inverse of a, and write it $a^{-1} \bmod N$
  - $\mathbb{Z}_N^*$ under multiplication mod N form a group.

Order

$$\mathbb{Z}_N^* := \{a \in \mathbb{Z}_N : \gcd(a, N) = 1\}, \varphi(N) = |\mathbb{Z}_N^*|$$

- Def (order mod N). Given $a \in \mathbb{Z}_N^*$, $\mathrm{ord_N}(a) := \min\{r : a^r \equiv 1 \bmod N\}$
- Fact (Euler's Theorem). $\forall\, a \in \mathbb{Z}_N^*, a^{\varphi(N)} \equiv 1 \bmod N$
  - →$\mathrm{ord}_N(a)$ is well-defined
  - →$\mathrm{ord}_N(a) \mid \varphi(N)$

Exercises

$\varphi(p) = p-1$

$\varphi(p \cdot q) = \varphi(p) \cdot \varphi(q)$

- Show that $\mathrm{ord}_N(a) \mid \varphi(N)$ always holds.
- Let $a = 4, N = 35$
  - $\mathbb{Z}_{35}^* = \{1, 2, 3, 4, 5, \ldots\}$ 9 $(5, 35) = 5$
  - $\mathrm{ord}_{35}(4) = 4^1 = 4, 4^2 = 16, 4^3 = 29, 4^4 = 11, 4^5 = 9, 4^6 = 1$ (mod 35)

$\mathrm{ord}_{35}(4) = 6$

$4^4 = 6$ / 11

$\varphi(35) = |\mathbb{Z}_{35}^*| = 24$ (33)

$= \varphi(5 \times 7) = \varphi(5) \cdot \varphi(7)$

$= 4 \times 6 = 24$

Order finding

Order finding

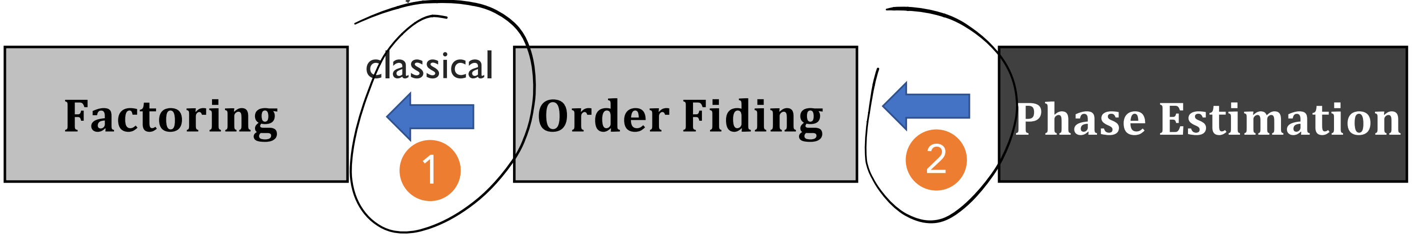
Input. Positive integer $N \geq 2, a \in \mathbb{Z}_N^*$

Goal. Compute $\mathrm{ord}_N(a)$

- Theorm. Factorization $\equiv$ Order finding
  - We can solve one efficiently iff. we can solve the other efficiently.

  → Best classical algorithm takes exponential time

- Theorm. ∃ poly-time quantum algorithm for order finding (hence factorization too)



Reducing factoring to order finding

Input. $N (= pq)$
Goal. Find p, q

Factoring

$a \rightarrow$ **Order Fiding** $\rightarrow ord_N(a)$

Given

- Idea: Pick random $a \in \mathbb{Z}_N^*$, compute $\underline{r = ord_N(a)}$

$$a^r \equiv 1 \bmod N \Leftrightarrow N |\, a^r - 1$$

- If r happens to be even, $a^r - 1 = (a^{\frac{r}{2}}+1)(a^{\frac{r}{2}}-1)$

$$p \cdot q = N | \underbrace{(a^{\frac{r}{2}}+1)}_{A}\underbrace{(a^{\frac{r}{2}}-1)}_{B}$$

$pq \nmid A,\ pq \nmid B$

$c \cdot pq = A \cdot B$

- ~~can $N | (a^{\frac{r}{2}}-1)$?~~ $a^{r/2} \equiv 1 \bmod N$. $r/2 < r$, contradicts def. of order
- What if $\underline{N \nmid (a^{\frac{r}{2}}+1)}$?

could happen. assume it doesn't.

$\gcd(a^{r/2}-1, N) = p\ (q)$

Reducing factoring to order finding cont'd

Input: an odd, composite integer N that is not a prime power.

Repeat
  Randomly choose $a \in \{2, \ldots, N-1\}$.
  Compute $d = \gcd(a, N)$.
  If $d \geq 2$ then /* We've been incredibly lucky. */
    Return $u = d$ and $v = N/d$.
  Else /* Now we know $a \in \mathbb{Z}_N^*$. */
    Let r be the order of a in $\mathbb{Z}_N^*$. /* Requires the order finding algorithm. */
    If r is even then
      Compute $x = a^{r/2} - 1 \pmod N$.
      Compute $d = \gcd(x, N)$.
      If $d \geq 2$ then
        Return $u = d$ and $v = N/d$. /* Answer is found. */
Until answer is found (or you get tired).

efficient

- Bad a
  - $ord_N(a)$ is odd
  - $N|(a^{\frac{r}{2}}+1)$
- Fact. $\Pr_{a \leftarrow \mathbb{Z}_N^*}[a \text{ BAD}] \leq \frac{1}{2}$

➔Succeed in k iterations with prob. $\geq 1 - \frac{1}{2^k}$.

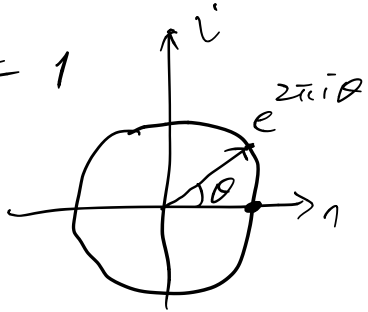
- Runtime = $O(k \cdot \text{Order-finding})$

Exercise

Let $\omega_r := e^{2\pi i \frac{1}{r}}$ be the r^{th} root of unity

1. Show that $\omega_r^r = 1$.

$$\omega_r^r = \left(e^{2\pi i \frac{1}{r}}\right)^r = e^{2\pi i \cdot 1} = 1$$



2. Show that $\sum_{j=0}^{r-1} \omega_r^j = 0$.

$$\sum_{j=0}^{r-1} \omega_r^j = \underbrace{\omega_r^0}_{=1} + \omega_r^1 + \dots + \omega_r^{r-1}$$

$$= \frac{1 - \omega_r^r}{1 - \omega_r} \quad \omega_r^r = 1$$

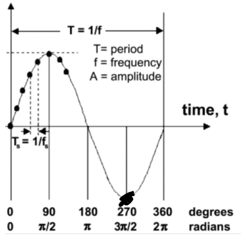
$$= 0$$

$$1 + x + x^2 + \dots + x^{k-1} = \frac{1 - x^k}{1 - x}$$

Phase estimation

Meaning of "phase"

- Phase transition
  - Solid → liquid → gas , plasma
  - https://en.wikipedia.org/wiki/Phase_transition
- Phase in periodic function (waves)
  - Location with a single wave length
  - https://en.wikipedia.org/wiki/Phase_(waves)



- Phase factor $e^{i\theta}$
  - Global phase: $e^{i\theta}|\psi\rangle$ vs. $|\psi\rangle$ same statistics under measurements
  - Relative phase: $|0\rangle + e^{i\theta}|1\rangle$
    - $|0\rangle + |1\rangle$ vs. $|0\rangle - |1\rangle$: Measurement statistics differ

$\{|+\rangle, |-\rangle\}$

Phase estimation (a.k.a. eigenvalue est.)

Input:

- Unitary operation U (described by a quantum circuit).
- A quantum state $|\psi\rangle$ that is an eigenvector of U: $U|\psi\rangle = e^{2\pi i\theta}|\psi\rangle$.

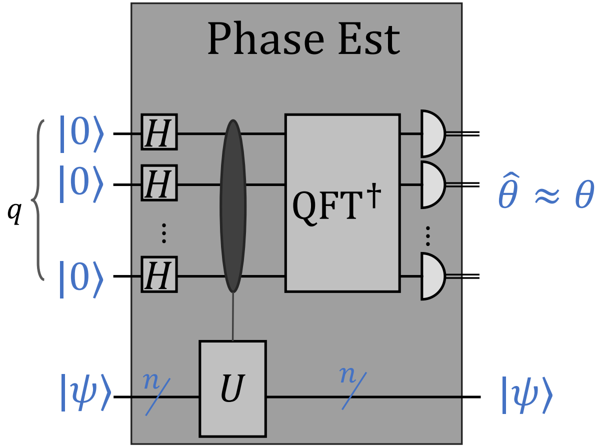
Output: An approximation to $\theta \in [0,1)$.

- Fact (HW4): Unitary U on n qubits has a complete set of orthonormal eigenvectors $\{|\psi_1\rangle, |\psi_2\rangle, \dots, |\psi_N\rangle\}, N = 2^n$
  - $\langle\psi_j|\psi_k\rangle = \begin{cases} 1, j = k \\ 0, j \neq k \end{cases}$
  - $U|\psi_j\rangle = e^{2\pi i\theta_j}|\psi_j\rangle$

$\lambda_j = e^{2\pi i\theta_j}$

$|\lambda_j| = 1$

Kitaev's quantum phase estimation algorithm



- Theorem. PE produces $\hat{\theta}$ with
  - precision $|\hat{\theta} - \theta| \leq \delta$ and
  - failure probability $\leq \varepsilon$

whenever $t = \Omega(\log \frac{1}{\delta \cdot \varepsilon})$.

- Proof (Next time)

~~Theorem. PE produces $\hat{\theta}$ with~~

Solving order finding by phase estimation

Reducing order finding to phase estimation

Given $a \in \mathbb{Z}_N^*$, find $r := ord_N(a)$.

[$n \sim \log N$: # bits to encode elements of $\mathbb{Z}_N^*$]

- Wishful thinking:
  - A unitary operation U easy to implement
  - An eigenvector $|\psi\rangle$ whose eigenvalue reveals r. (Ex. $U\,|\psi\rangle = e^{2\pi i\theta}|\psi\rangle, \theta = 1/r$)
  - Plug into Phase Estimation and done!

A proper unitary and eigenvec for order finding

Given $a \in \mathbb{Z}_N^*$, find $r := ord_N(a)$.

$M_{a^{-1}}: |x\rangle \mapsto |a^{-1}x \bmod N\rangle$

[$n \sim \log N$: # bits to encode elements of $\mathbb{Z}_N^*$]

$\omega_r := e^{2\pi i \frac{1}{r}}$ (r^{th} root of unity)

$\omega_r^r := e^{2\pi i \frac{r}{r}} = 1$

- Unitary M_a: $|x\rangle \mapsto |ax \bmod N\rangle$

$x \in \{0,1\}^n \quad N < 2^n \quad N \le x < 2^n$

$|x\rangle \longmapsto |x\rangle$

? - Eigenvector: $|\psi\rangle := \frac{1}{\sqrt{r}}(|1\rangle + \omega_r^{-1}|a\rangle + \omega_r^{-2}|a^2\rangle + \cdots + \omega_r^{-(r-1)}|a^{r-1}\rangle)$

$$M_a|\psi\rangle = \frac{1}{\sqrt{r}}\left(M_a|1\rangle + \omega_r^{-1} M_a|a\rangle + \cdots + \omega_r^{-(r-1)}|a^{r-1}\rangle\right)$$

mod N

$$= \frac{1}{\sqrt{r}}\left(|a\rangle + \omega_r^{-1}|a^2\rangle + \cdots + \omega_r^{-(r-1)}|a^r\rangle\right)$$

$\omega_r^r = \omega_r^{-r} = 1$

$$= \frac{1}{\sqrt{r}}\left(\omega_r^{-(r-1)}|1\rangle + |a\rangle + \omega_r^{-1}|a^2\rangle + \cdots\right)$$

- What is missing?

don't know r. how to prepare $|\psi\rangle$

$$= \omega_r|\psi\rangle = e^{2\pi i \frac{1}{r}} \cdot |\psi\rangle$$

Live with a set of eigenvectors

- Unitary $M_a: |x\rangle \mapsto |ax \bmod N\rangle$

$$\omega_r := e^{2\pi i \frac{1}{r}}$$

- Eigenvector: $|\psi\rangle := \frac{1}{\sqrt{r}}(|1\rangle + \omega_r^{-1}|a\rangle + \omega_r^{-2}|a^2\rangle + \cdots + \omega_r^{-(r-1)}|a^{r-1}\rangle)$

$$|\psi_0\rangle = 1/\sqrt{r}(|1\rangle + |a\rangle + |a^2\rangle + \cdots + |a^{r-1}\rangle)$$

ω_r^0

$$|\psi_1\rangle = 1/\sqrt{r}(|1\rangle + \omega_r^{-1}|a\rangle + \omega_r^{-2}|a^2\rangle + \cdots + \omega_r^{-(r-1)}|a^{r-1}\rangle) = |\psi\rangle$$

$M_a|\psi_1\rangle = \omega_r|\psi_1\rangle$

$$\vdots$$

$$|\psi_j\rangle = \frac{1}{\sqrt{r}}(|1\rangle + \omega_r^{-j}|a\rangle + \omega_r^{-2j}|a^2\rangle + \cdots + \omega_r^{-(r-1)j}|a^{r-1}\rangle)$$

$$\vdots$$

$$|\psi_{r-1}\rangle = 1/\sqrt{r}\left(|1\rangle + \omega_r^{-(r-1)}\middle|a\right\rangle + \omega_r^{-2(r-1)}|a^2\rangle + \cdots + \omega_r^{-(r-1)(r-1)}|a^{r-1}\rangle)$$

Live with a set of eigenvectors cont'd

$\omega_r := e^{2\pi i \frac{1}{r}}$

- Unitary $M_a: |x\rangle \mapsto |ax \bmod N\rangle$
- Eigenvector: $|\psi\rangle := \frac{1}{\sqrt{r}}(|1\rangle + \omega_r^{-1}|a\rangle + \omega_r^{-2}|a^2\rangle + \cdots + \omega_r^{-(r-1)}|a^{r-1}\rangle)$

$$|\psi_0\rangle = \frac{1}{\sqrt{r}}(|1\rangle + |a\rangle + |a^2\rangle + \cdots + |a^{r-1}\rangle)$$

$$M_a|\psi_0\rangle = \frac{1}{\sqrt{r}}(M_a|1\rangle + M_a|a\rangle + \cdots + M_a|a^{r-1}\rangle)$$

$$= \frac{1}{\sqrt{r}}(|a\rangle + |a^2\rangle + \cdots + |\underbrace{a^r}_{=1}\rangle)$$

$$= \frac{1}{\sqrt{r}}(|1\rangle + |a\rangle + |a^2\rangle + \cdots + |a^{r-1}\rangle)$$

$$= 1 \cdot |\psi_0\rangle$$

Live with a set of eigenvectors cont'd

- Unitary $M_a: |x\rangle \mapsto |ax \bmod N\rangle$

$$\omega_r := e^{2\pi i \frac{1}{r}}$$

- Eigenvector: $|\psi\rangle := \frac{1}{\sqrt{r}}(|1\rangle + \omega_r^{-1}|a\rangle + \omega_r^{-2}|a^2\rangle + \cdots + \omega_r^{-(r-1)}|a^{r-1}\rangle)$

$$|\psi_j\rangle = \frac{1}{\sqrt{r}}(|1\rangle + \omega_r^{-j}|a\rangle + \omega_r^{-2j}|a^2\rangle + \cdots + \omega_r^{-(r-1)j}|a^{r-1}\rangle)$$

$$M_a|\psi_j\rangle = \frac{1}{\sqrt{r}}(M_a|1\rangle + \omega_r^{-j} M_a|a\rangle + \cdots + \omega_r^{-(r-1)\cdot j} M_a|a^{r-1}\rangle)$$

$$= \frac{1}{\sqrt{r}}(|a\rangle + \omega_r^{-j}|a^2\rangle + \cdots + \omega_r^{-(r-1)j}\underbrace{|a^r\rangle}_{=1})$$

$$= \frac{1}{\sqrt{r}}(\omega_r^{-(r-1)\cdot j}|1\rangle + |a\rangle + \omega_r^{-j}|a^2\rangle + \cdots)$$

$$= \omega_r^{j}|\psi_j\rangle = e^{2\pi i \frac{j}{r}}|\psi_j\rangle$$

Live with a set of eigenvectors cont'd

- Unitary $M_a: |x\rangle \mapsto |ax \bmod N\rangle$

$$\omega_r := e^{2\pi i \frac{1}{r}}$$

- Eigenvector: $|\psi\rangle := \frac{1}{\sqrt{r}}\left(|1\rangle + \omega_r^{-1}|a\rangle + \omega_r^{-2}|a^2\rangle + \cdots + \omega_r^{-(r-1)}|a^{r-1}\rangle\right)$

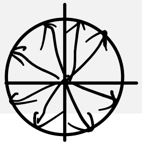
$$\left\{\begin{array}{l}
|\psi_0\rangle = 1/\sqrt{r}\left(|1\rangle + |a\rangle + |a^2\rangle + \cdots + |a^{r-1}\rangle\right) \\
\quad\vdots \\
|\psi_j\rangle = 1/\sqrt{r}\left(|1\rangle + \omega_r^{-j}|a\rangle + \omega_r^{-2j}|a^2\rangle + \cdots + \omega_r^{-(r-1)j}|a^{r-1}\rangle\right) \\
\quad\vdots \\
|\psi_{r-1}\rangle = 1/\sqrt{r}\left(|1\rangle + \omega_r^{-(r-1)}|a\rangle + \omega_r^{-2(r-1)}|a^2\rangle + \cdots + \omega_r^{-(r-1)(r-1)}|a^{r-1}\rangle\right)
\end{array}\right.$$

$$\sum_{j=0}^{r-1}|\psi_j\rangle = \frac{1}{\sqrt{r}}\left(r|1\rangle + \underbrace{\boxed{\sum_{k=0}^{r-1}(\omega_r^{-1})^k}}_{=0}|a\rangle + \underbrace{\boxed{\sum_{k=0}^{r-1}(\omega_r^{-2})^k}}_{=0}|a^2\rangle + \cdots + \underbrace{\boxed{\sum_{k=0}^{r-1}(\omega_r^{-(r-1)})^k}}_{=0}|a^{r-1}\rangle\right)$$

$= \sqrt{r}\,|1\rangle$

(ω_r^j)

$j = 0, \cdots r-1$



$\omega_r^{-(r-1)} = \omega_r$

$\sum_{k=0}^{r-1}(\omega_r)^k = 0$

Quantum order finding algorithm

$$|1\rangle = \frac{1}{\sqrt{r}} \sum_j |\psi_j\rangle, \; M_a|\psi_j\rangle = e^{2\pi i \frac{j}{r}}$$

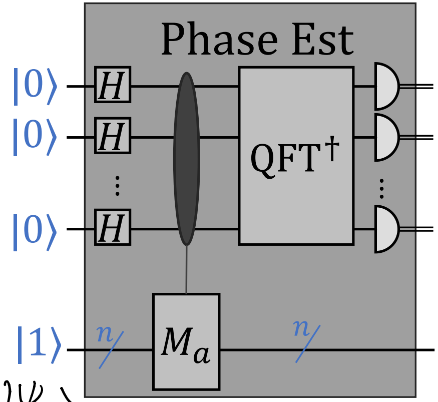
- Observation: $|\psi_j\rangle$ orthonormal

$$\langle \psi_j | \psi_k \rangle = \delta_{jk} \leftarrow \text{Ex.}$$

➔ PE with input $|1\rangle$

$\approx$ PE with $|\psi_j\rangle$ for a random j

- Post-processing to recover r



$|\psi_j\rangle$

Quantum order-finding algorithm

$1/r \quad j=1$

$\approx j/r$
for a random j

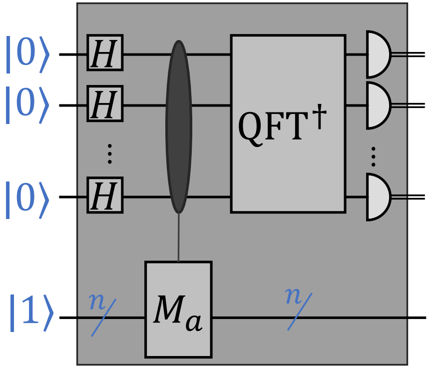
$$\ldots, \frac{j_1}{r}, \frac{j_2}{r}, \ldots \longrightarrow r$$

Summary

Factoring ← classical ① **Order Fiding** ← Quantum ② **Phase Estimation**

$$N|(a^{\frac{r}{2}}+1)(a^{\frac{r}{2}}-1)$$

- What's next?
  - Phase estimation algorithm
  - Complexity of quantum order finding (implementing controlled M_a)



Quantum order-finding algorithm

Logistics

- Proposal due Sunday May 3rd , 11:59pm AoE
  - Submit as a group via Gradescope
  - No group? Submit a proposal and I will coordinate
  - 1-2 pages: consisting of 1) the topic, background, context, and motivation; 2) identify a few core references; and 3) a goal you intend to achieve and a plan.
- Talk by Silverman in Math department
  - Cryptography and quantum computing
  - See campuswire for details. Register by May 6
- IBM Qiskit competition

Scratch