CSCE 440/640 Quantum Algorithms

Homework 1

Texas A&M U, Spring 2019 — *Jan. 14, 2019*

Lecturer: Fang Song — *Due: Jan. 28, 2019*

Instructions. Only PDF format is accepted (type it or scan clearly). Your solutions will be graded on *correctness* and *clarity*. You should only submit work that you believe to be correct; if you cannot solve a problem completely, you will get significantly more partial credit if you clearly identify the gap(s) in your solution. For this problem set, a random subset of problems will be graded. Problems marked with "[**G**]" are required for graduate students. Undergraduate students will get bonus points for solving them.

You may collaborate with others on this problem set. However, you must ***write up your own solutions*** and ***list your collaborators*** for each problem.

The review materials on Piazza/Resources/Math Review (Week 1) might be helpful.

1. **Attention: this problem is due Sunday, January 20, 11:59pm CST.**
   (a) (2 points) Enroll on Piazza `https://piazza.com/tamu/spring2019/csce440640/`.
   (b) (3 points) Post a note on Piazza describing: 1) a few words about yourself; 2) your strengths in CS (e.g., programming, algorithm, ...); 3) what you hope to get out of this course; and 4) anything else you feel like sharing. See instructions on how to post a note `https://support.piazza.com/customer/en/portal/articles/1564004-post-a-note`.

   The purpose is to help me know you all, and also get you known to your fellow students. You can follow up the posts and start looking for group members for your course project.
   (c) (1 point (bonus)) Upload a profile picture.

2. (Basic algebra) Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Y = i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (Physicists call them the Pauli operators).
   (a) (6 points) (Complex number) Let $c = a + bi$ be a complex number. The real and imaginary parts of c are denoted $Re(c) = a$ and $Im(c) = b$.
   - Prove that $c + c^* = 2 \cdot Re(c)$.
   - Prove that $cc^* = a^2 + b^2$. (NB. $|c| := \sqrt{cc^*}$ is called the magnitude of c. Can you see the meaning of $|c|$ on the complex plane?)
   - What is the polar form of $c = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$? Use the fact that $e^{i\theta} = \cos\theta + i\sin\theta$?

   (b) (10 points) (Dirac notation)

- Write $\frac{1}{2}|0\rangle + \frac{1+\sqrt{2}i}{2}|1\rangle$ in column vector form.
- Find the eigenvalues and the corresponding eigenvectors of each Pauli operator (except I). Express the eigenvectors using the Dirac notation.

(c) (4 points) (Basis) We know that $\{|0\rangle, |1\rangle\}$ forms an orthonormal basis of $\mathbb{C}^2$, which is usually called the computational basis. Consider

$$|+\rangle := \frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle; \quad |-\rangle := \frac{1}{2}|0\rangle - \frac{1}{2}|1\rangle .$$

Prove that $\{|+\rangle, |-\rangle\}$ also forms an orthonormal basis. (usually called the Hadamard basis)

(d) (12 points) (Trace) Recall the trace of a square matrix $M = (m_{ij})_{n\times n}, m_{ij} \in \mathbb{C}$ is defined by $\mathrm{tr}(M) := \sum_{i=1}^{n} m_{ii}$.

- What is $\mathrm{tr}(X|0\rangle\langle 1|)$?
- Show that $\mathrm{tr}(YZ) = \mathrm{tr}(ZY)$. Prove that this holds for general matrices: any $n \times n$ matrices M and N, $\mathrm{tr}(MN) = \mathrm{tr}(NM)$.
- Prove that the trace has the cyclic property $\mathrm{tr}(ABC) = \mathrm{tr}(BCA)$.

(e) (8 points) (Inner/outer product)

- Let $|\phi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$. $|\psi\rangle = \sqrt{\frac{1}{3}}|0\rangle - \sqrt{\frac{2}{3}}|1\rangle$. Calculate $\langle\phi|\psi\rangle$.
- Show that $X = |0\rangle\langle 1| + |1\rangle\langle 0|$. Express Y, Z in this outer product form too. Calculate $\langle 1|X|0\rangle$ (using linearity).

3. (Probability) Let X, Y be random variables. $\mathrm{E}[\cdot]$ denotes expectation.

(a) (3 points) Give an example of two random variables X, Y such that $\mathrm{E}[XY] \neq \mathrm{E}[X]\mathrm{E}[Y]$.

(b) (5 points) Let X be the number of `HEADS` after n fair coin tosses. Calculate $\mathrm{E}[X]$. (Hint: use linearity of expectation)

(c) (4 points) Let X be as above. Show that $\Pr[X \geq 0.6n] \leq 0.85$.

(d) (6 points) Let Y be the outcome of a *biased* coin toss, which lands on `HEADS` with probability 0.1. Suppose one makes multiple tosses independently. What is the probability that one sees a `HEADS` at least once after t tosses? Derive the minimum number t so that this probability is greater than 0.99.

4. (10 points) (Birthday bound) Fix a positive integer N, and $q \leq \sqrt{2N}$. Choose elements $y_1, \ldots, y_q$ uniformly and independently at random from a set of size N. Show that the probability that there exist distinct i, j with $y_i = y_j$ is $\Theta(q^2/N)$. (Note: you need to prove both lower and upper bounds.)

5. (8 points) (Asymptotic notations) In each of the following, answer True or false.

(a) $1000n^2 = O(.00001n^3)$.

(b) $5\log^{23} n = o(n)$.

(c) $e^n = \Omega(2^{n^2})$.

(d) $100n^3 + 10n + 1 = \omega(n^{3.001})$.

6. (12 points) (Quantum states and gates) Let $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $H' = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$.
   In each case, describe the resulting state.
   (a) Apply H to the qubit $\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$.
   (b) Apply H to the first qubit of state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.
   (c) Apply H to both qubits of $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.
   (d) Apply H' to both qubits of state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.