M, 11/18/19

Fall'19 CSCE 629

Analysis of Algorithms

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Lecture 31

- Approximating vertex cover
- Linear programming relaxation

Coping with NP-Completeness

"I can't find an efficient algorithm, I guess I'm just too dumb."

"I can't find an efficient algorithm, but neither can all these famous people."

https://i.stack.imgur.com/EkpIV.jpg

- Better (constructive) answers: sacrifice one of three desired features
  1. ~~Solve arbitrary instances~~
  2. ~~Solve problems in poly-time~~
  3. ~~Solve problems to optimality~~
- Techniques
  - Identifying structured special cases
  - Local search heuristics (e.g., gradient descent)
  - Approximation algorithms

Finding near-optimal vertex cover

Input. Graph $G = (V, E)$

- Vertex cover $S \subseteq V$: subset of vertices that touches all edges

Goal. Find a vertex cover S of minimum size

- First attempt: greedily pick the vertex that covers most edges

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APP-VC: on input G = (V, E)
  For v ∈ V (in descending order of degrees)
      Add v in S
      Delete v and its neighbors from G
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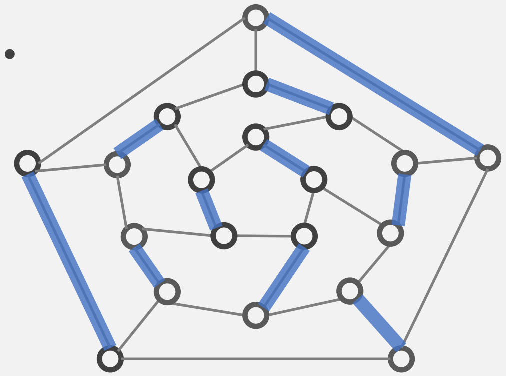
- Claim. Suppose the minimum vertex cover has size OPT. Then the output of APP-VC has size at most $O(\log n \cdot \mathrm{OPT})$
- Pf. Exercise (Hint on board)

2-approximation vertex cover

Recall: $M \subseteq E$ is a matching in $G = (V, E)$ if each node appears in at most one edge in M.

Observation: For any matching M and any vertex cover S, $|M| \leq |S|$. In particular, $|M| \leq \mathrm{OPT}$ (size of min vertex cover).

- 2nd attempt: find a MAX matching



> 2-APP-VC: on input $G = (V, E)$
> Find a maximal matching $M \subseteq E$
> Return $S = \{\text{all end points of edges in } M\}$

- Claim. The output of 2-APP-VC has size at most $2 \cdot \mathrm{OPT}$
- Pf. $|S| = 2|M| \leq 2 \cdot \mathrm{OPT}$. Why does S have to be a vertex cover?
  - Exercise. Is this tight, i.e., 2-APP-VC's output $= 2 \cdot \mathrm{OPT}$ on some graph?

Integer linear programming (ILP)

Input. Graph $G = (V, E)$

- Vertex cover $S \subseteq V$: subset of vertices that touches all edges

Goal. Find a vertex cover S of minimum size

- Formulating vertex cover as an integral linear program

For each $i \in V$, introduce $x_i \in \{0,1\}$
Min $\sum_{i=1}^{n} x_i$
Subject to:
$$x_i + x_j \geq 1 \quad \text{for each } (i,j) \in E$$

[i.e., Pick i in vertex cover iff. $x_i = 1$]

☹ We don't know (expect) a poly-time algorithm (ILP)

- Without integrality (LP), we do know poly-time algorithms

Putting aside the integral constraint

(ILP Π) Min $\sum_{i=1}^{n} x_i$
Subject to:
$$x_i + x_j \geq 1, \qquad \forall (i,j) \in E$$
$$x_i \in \{0,1\}, \qquad \forall i \in V$$

(LP Σ) Min $\sum_{i=1}^{n} x_i$
Subject to:
$$x_i + x_j \geq 1, \qquad \forall (i,j) \in E$$
$$0 \leq x_i \leq 1, \qquad \forall i \in V$$

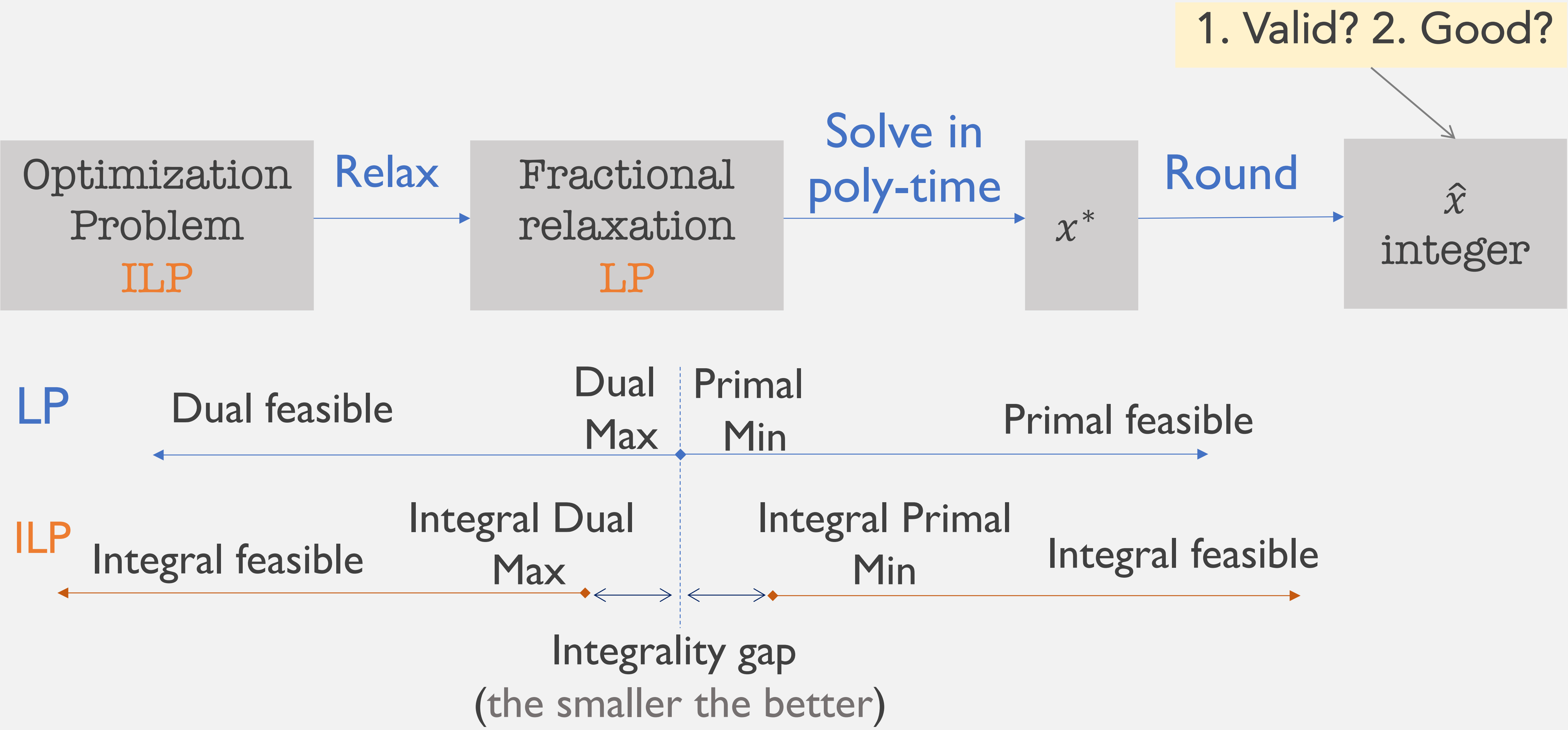
?

Let x^* be an optimal soln. for LP Σ
& optimal value $\mathrm{OPT} = \sum_i x_i^*$

$$x_i := \lfloor x_i^* \rceil = \begin{cases} 1, & \text{if } x_i^* \geq \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

- (Threshold) Rounding:
  i. $\{x_i\}$ is a feasible integral solution: $\forall (i,j) \in E, x_i^* \geq \frac{1}{2}$ or $x_j^* \geq \frac{1}{2}$ or both
  ii. $\sum_i x_i \leq \sum_i 2 \cdot x_i^* = 2 \cdot \mathrm{OPT} \leq 2 \cdot \mathrm{OPT}_{\mathrm{Int}}$ [optimal value of ILP Π, i.e. size of min vertex cover]

LP relaxation



Approximating set cover

Input. Set U of n elements, $S_1, \dots, S_m$ of subsets of U

Goal. Find $I \subseteq \{1, \dots, m\}$ of minimum size such that $\bigcup_{i \in I} S_i = U$

(ILP Π for Set cover)

For each $i \in \{1, \dots, m\}$, introduce $x_i \in \{0,1\}$

Min $\sum_{i=1}^{m} x_i$

Subject to:

$$\sum_{i : u \in S_i} x_i \geq 1, \qquad \forall u \in U$$

LP relaxation for set cover

(Set cover ILP Π)
Min $\sum_{i=1}^{m} x_i$
Subject to:
$$\sum_{i:u\in S_i} x_i \geq 1, \qquad \forall u \in U$$
$$x_i \in \{0,1\}, \qquad \forall i \in \{1, \dots, m\}$$

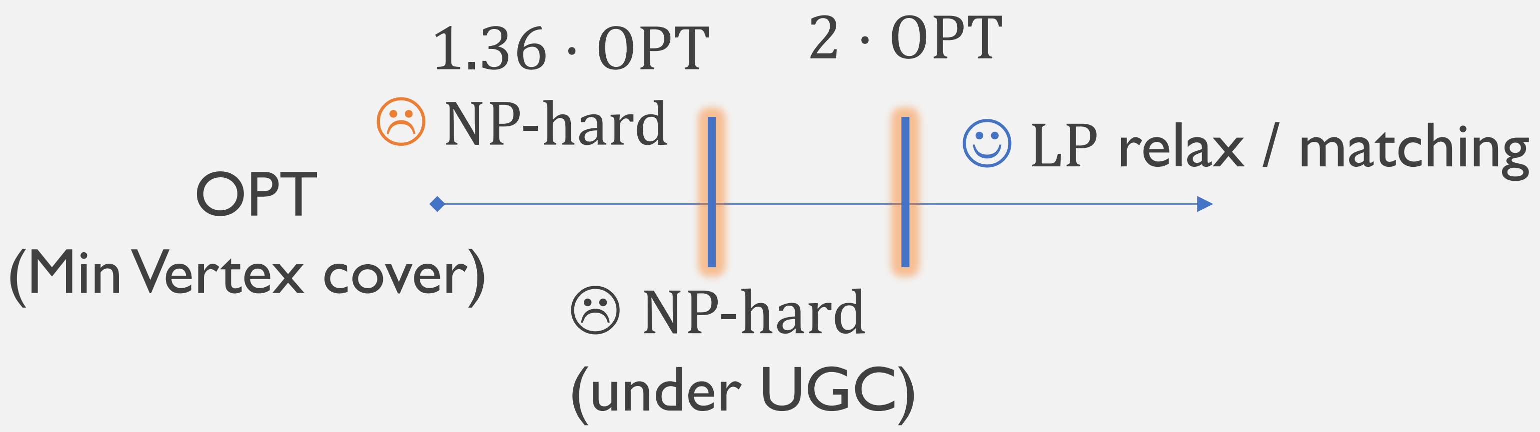
(Set cover Σ)
Min $\sum_{i=1}^{m} x_i$
Subject to:
$$\sum_{i:u\in S_i} x_i \geq 1, \qquad \forall u \in U$$
$$0 \leq x_i \leq 1, \forall i \in \{1, \dots, m\}$$

Let x^* be an optimal soln. for LP Σ
& optimal value $\mathrm{OPT} = \sum_i x_i^*$

? $x_i := \lfloor x_i^* \rceil$

- Threshold rounding: does it cover all elements?
  - Ex. $u \in S_1, \dots, S_{100}; x_1^*, \dots x_{100}^* = \frac{1}{100} \Rightarrow x_1 = \cdots = x_{100} = 0$. u is missed!
- Randomized rounding! [Stay tuned]

Hardness of approximation



Theorem. It is NP-Hard to approximate Vertex Cover to with any factor below 1.36067. [i.e., otherwise, you can solve 3-SAT in poly-time]

Theorem’. It is NP-Hard to approximate Vertex Cover to with any factor below 2, assuming the unique games conjecture (UGC).

Want to read more?
https://cs.nyu.edu/~khot/papers/UGCSurvey.pdf
https://cs.stanford.edu/people/trevisan/pubs/inapprox.pdf