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Fall'19 CSCE 629

Analysis of Algorithms

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Lecture 24

- Linear programming

Credit: based on slides by K. Wayne

Linear programming

■ "Standard form" of an LP

- $m = \#$ constraints, $n = \#$ decision variables. $i = 1, \dots, m, j = 1, \dots, n$
- **Input:** real numbers c_j, a_{ij}, b_i
- **Output:** real numbers x_j
- Maximize linear objective function subject to linear inequalities
- Feasible vs. optimal soln's.

$$\begin{aligned} &\text{Max } \sum_{j=1}^n c_j x_j \\ &\text{Subject to: // linear constraints} \\ &\quad \sum_{j=1}^n a_{ij} x_j \leq b_i \quad 1 \leq i \leq m \\ &\quad x_j \geq 0 \quad 1 \leq j \leq n \end{aligned}$$

\equiv

$$\begin{aligned} &\text{Max } \mathbf{c}^T \mathbf{x} \\ &\text{Subject to: } \mathbf{Ax} \leq \mathbf{b} \\ &\quad \mathbf{x} \geq \mathbf{0} \end{aligned}$$

$$\mathbf{A} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \dots & \dots & \dots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$$

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \dots \\ c_n \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} \quad \mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \end{pmatrix}$$

Linear programming: variants

- “Slack form” of an LP: linear equalities

$$\begin{aligned} &\text{Max } \sum_{j=1}^n c_j x_j \\ &\text{Subject to: // linear constraints} \\ &\quad \sum_{j=1}^n a_{ij} x_j \leq b_i \quad 1 \leq i \leq m \\ &\quad x_j \geq 0 \quad 1 \leq j \leq n \end{aligned}$$

⇒

$$\begin{aligned} &\text{Max } \sum_{j=1}^n c_j x_j \\ &\text{Subject to: // linear constraints} \\ &\quad s_i = b_i - \sum_{j=1}^n a_{ij} x_j \quad 1 \leq i \leq m \\ &\quad (\text{slack vars}) s_i \geq 0 \quad 1 \leq i \leq m \\ &\quad x_j \geq 0 \quad 1 \leq j \leq m \end{aligned}$$

- Other equivalent variations
 - Minimization vs. maximization
 - Variables without **nonnegativity** constraints
 - \geq vs. \leq

Geometry of linear programming

1. Feasible

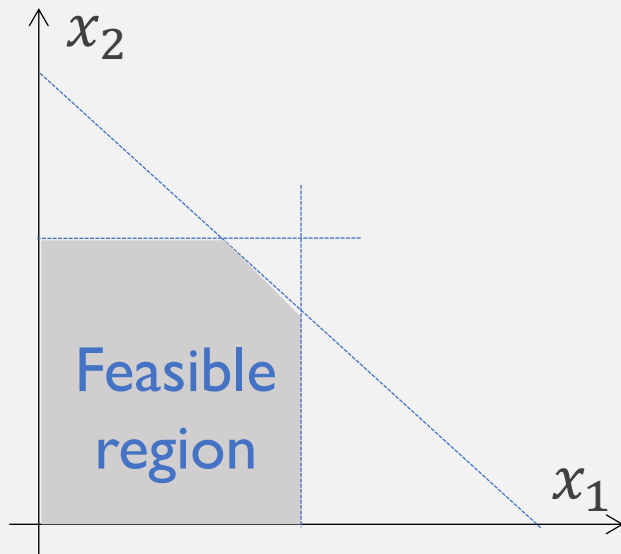
Maximize: $x_1 + 5x_2$

Subject to:

$$0 \leq x_1 \leq 12$$

$$0 \leq x_2 \leq 15$$

$$x_1 + x_2 \leq 24$$



2. Infeasible

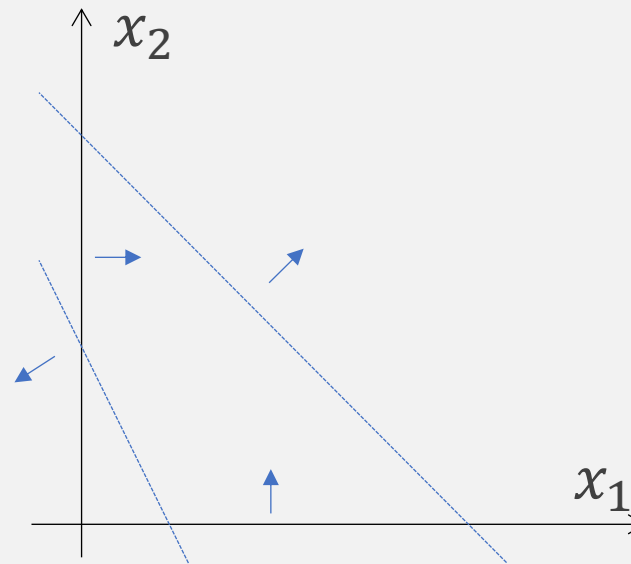
Maximize: $x_1 - x_2$

Subject to:

$$2x_1 + x_2 \leq 1$$

$$x_1 + x_2 \geq 2$$

$$x_1, x_2 \geq 0$$



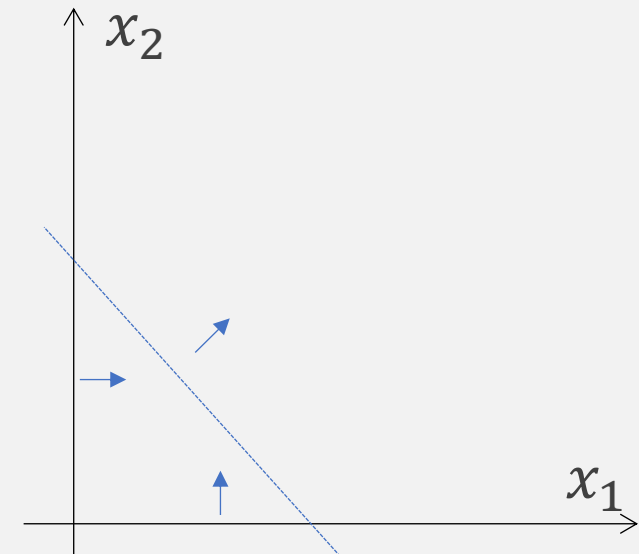
3. Unbounded

Maximize: $2x_1 + x_2$

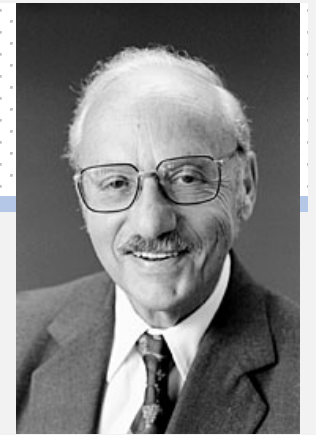
Subject to:

$$x_1 + x_2 \geq 1$$

$$x_1, x_2 \geq 0$$



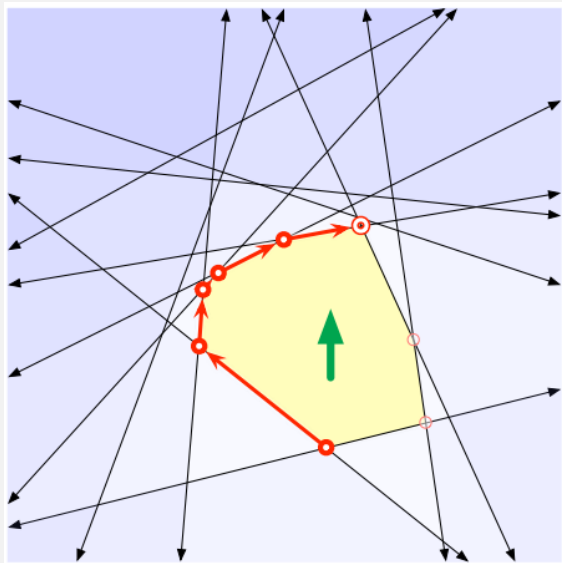
Simplex algorithm: the gist



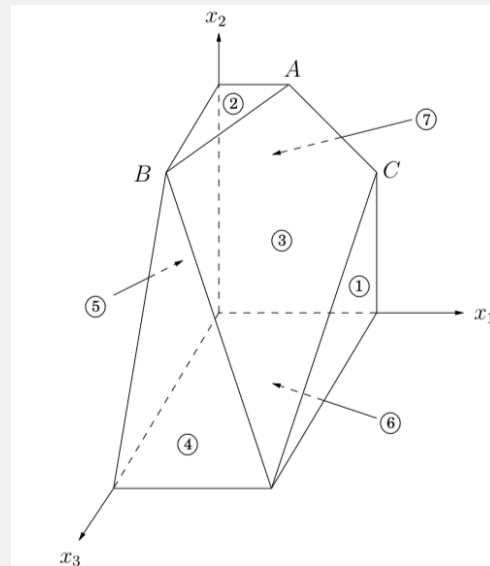
George Dantzig 1947

Let v be any **vertex** of the **feasible region**
While there is a **neighbor** u of v with better obj. value

$$v \leftarrow u$$



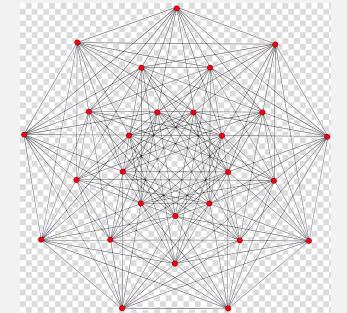
“Hill-climbing” along
vertices in the polygon



3D-polyhedron defined
by 7 inequalities

n variables?

- A linear eq. defines a **hyperplane** in R^n
- A linear ineq. defines a **halfspace** in R^n
- Each **vertex** is specified by n ineq's
- 2 vertices are **neighbors** if share $n - 1$ defining ineq's



Simplex algorithm: the fine prints

Let v be any **vertex** of the **feasible region**

While there is a **neighbor** u of v with better obj. value

$v \leftarrow u$

■ How to find an initial feasible vertex?

- Reduced to an LP and solved by simplex!

■ Which neighbor to move to? (Pivot)

■ Running time? [m ineq's, n variables]

☹ **Exponential** in worst-case! $\binom{m+n}{n} \geq \left(1 + \frac{m}{n}\right)^n$ vertices

☺ Super fast in real world [typically terminates after at most $2(m+n)$ pivots]

■ Correctness?

- Convex polyhedron & linear objective function: local max \equiv global max

Poly-time algorithms for linear programming

- Ellipsoid algorithm [Khachiyan 1979]

POLYNOMIAL ALGORITHMS IN LINEAR PROGRAMMING*

L. G. KHACHIYAN

Moscow

- A mathematical “Sputnik”
- Not competitive in practice

- Interior point algorithm [Karmarkar 1984]

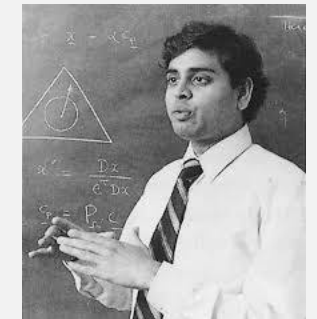
A New Polynomial-Time Algorithm for Linear Programming

N. Karmarkar

AT&T Bell Laboratories
Murray Hill, New Jersey 07974



Leonid Khachiyan



Narendra Karmarkar

N.B. Commercial solvers can solve LPs with millions of variables and tens of thousands of constraints.

How to decide optimality?

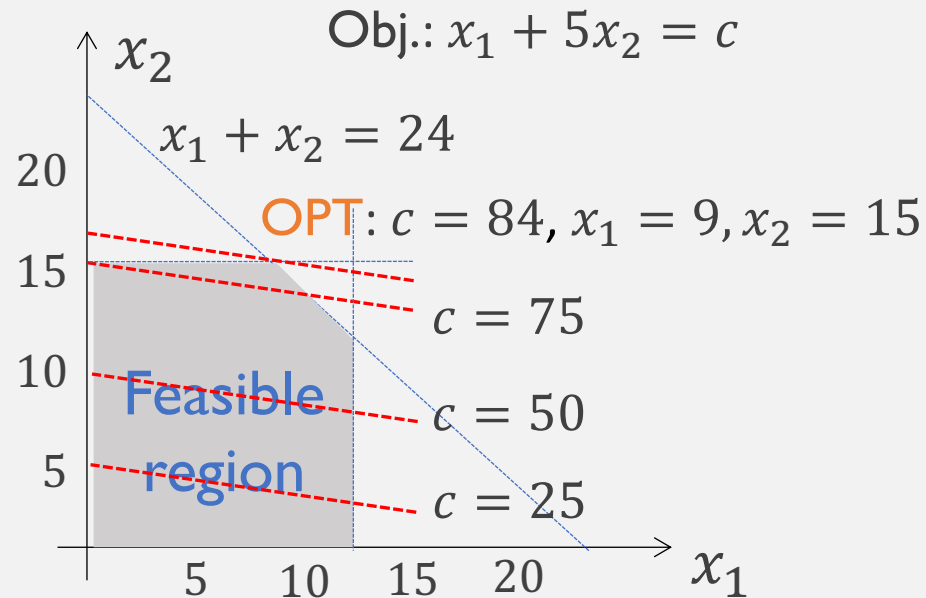
(P) Maximize: $x_1 + 5x_2$

Subject to:

$$0 \leq x_1 \leq 12$$

$$0 \leq x_2 \leq 15$$

$$x_1 + x_2 \leq 24$$



Certificate: $x_1 + 5x_2 = 4 \cdot x_2 + 1 \cdot (x_1 + x_2) \leq 4 \cdot 15 + 24 = 84$

How to find these (magic) multipliers?

Recall: max-flow & min-cut duality

- Weak duality (certificate of optimality)

$$v(f) \leq \text{cap}(A, B)$$

- Strong duality (max-flow min-cut theorem)

Value of max flow = capacity of min cut

Linear programming duality

(Primal) Max $\sum_{j=1}^n c_j x_j$

Subject to:

$$\sum_{j=1}^n a_{ij} x_j \leq b_i \quad 1 \leq i \leq m$$

$$x_j \geq 0 \quad 1 \leq j \leq n$$

(Dual) Min $\sum_{i=1}^m b_i y_i$

Subject to:

$$\sum_{i=1}^m a_{ij} y_i \geq c_j \quad 1 \leq j \leq n$$

$$y_i \geq 0 \quad 1 \leq i \leq m$$

(Primal) Max $c^T x$

Subject to:

$$Ax \leq b$$

$$x \geq 0$$

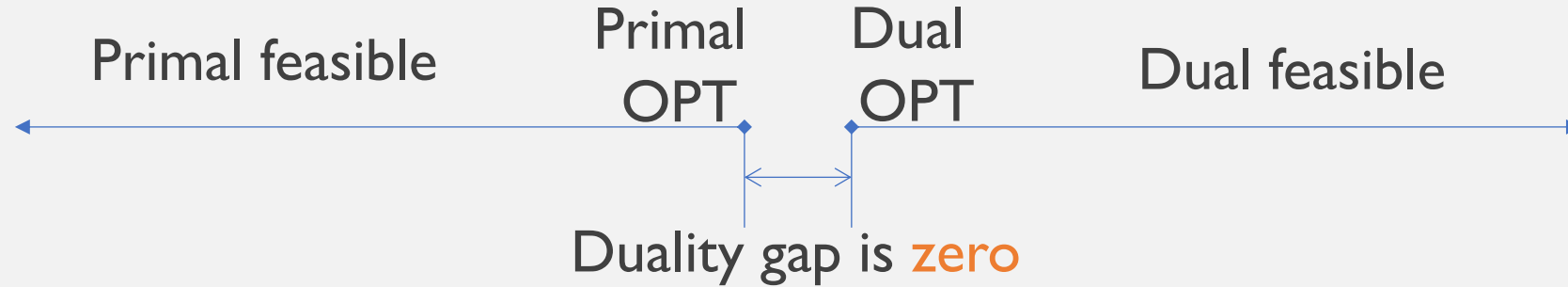
(Dual) Min $y^T b$

Subject to:

$$y^T A \geq c^T$$

$$y \geq 0$$

Fundamental theorem of linear programming



$$\begin{aligned} & \text{(Primal) Max } c^T x \\ & \text{Subject to:} \\ & \quad Ax \leq b \\ & \quad x \geq 0 \end{aligned}$$

$$\begin{aligned} & \text{(Dual) Min } y^T b \\ & \text{Subject to:} \\ & \quad y^T A \geq c^T \\ & \quad y \geq 0 \end{aligned}$$

- **Weak duality.** If x is a feasible solution for a linear program Π , and y is a feasible solution for its dual \square , then $c^T x \leq y^T Ax \leq y^T b$.
- **Strong duality.** Π has an optimal solution and x^* **if and only if** its dual \square has an optimal solution y^* such that $c^T x = y^T Ax = y^T b$.

Duality example

(P) Maximize: $x_1 + 5x_2$
Subject to:

$$0 \leq x_1 \leq 12$$

$$0 \leq x_2 \leq 15$$

$$x_1 + x_2 \leq 24$$

Max = 84, $x_1 = 9, x_2 = 15$

(D) Minimize: $12y_1 + 15y_2 + 24y_3$
Subject to:

$$y_1 + y_3 \geq 1$$

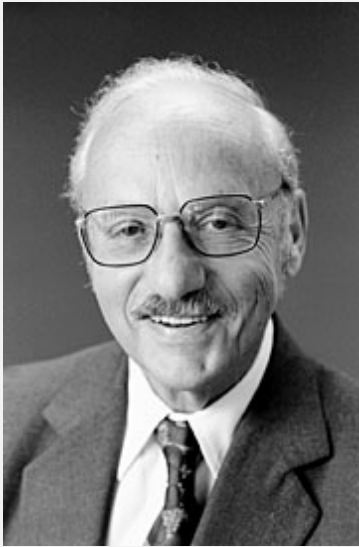
$$y_2 + y_3 \geq 5$$

$$y_1, y_2, y_3 \geq 0$$

Min = 84, $y_1 = 0, y_2 = 4, y_3 = 1$

(magic) multipliers

A dialogue between Dantzig & von Neumann



George Dantzig

Let me show you my exciting finding: simplex algorithm for LP
... .. [next 30 mins]

Get to the point, please!

OK! Em... To be concise ... [next 3 mins]

Ah, that!



John von Neumann

[next 60 mins]
.... (convexity)... (fixed point) ... (2-player game) ...
so, there is duality which'd follow by my **min-max theorem** ...

For any matrix A , $\min_x \max_y xAy = \max_y \min_x xAy$.

Exercise: Multicommodity flow

- A flow network with multiple flows (commodities)
 - $c(e)$: capacity on each edge
 - $K_i = (s_i, t_i, d_i)$: source, sink, and demand of commodity i . $i = 1, \dots, \ell$
- **Goal.** Decide if it is possible to accommodate all commodities

Max/min: 0

Subject to:

$$\begin{aligned} f_{ie} &\geq 0, & \forall e \in E \\ \sum_{i=1}^{\ell} f_{ie} &\leq c(e), & \forall e \in E \\ \sum_{e \text{ into } v} f_{ie} - \sum_{e \text{ out of } v} f_{ie} &= 0, & \forall v \in V \setminus \{s, t\} \\ \sum_{e \text{ out of } s_i} f_{ie} - \sum_{e \text{ into } s_i} f_{ie} &= d_i, & i = 1, \dots, \ell \end{aligned}$$