W, 10/30/19

Fall'19 CSCE 629

Analysis of Algorithms

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Lecture 23

- Remarks on Ford-Fulkerson
- Intro to linear programming

Credit: based on slides by A. Smith & K. Wayne

Ford-Fulkerson augmenting-path algorithm

Ford–Fulkerson(G, s, t, c)
 For each $e \in E$ $f(e) \leftarrow 0$, $G_f \leftarrow residual\ graph$
 While there is an augmenting path P in G_f
   $f \leftarrow Augment(f, c, P)$
   Update G_f
 return f

Theorem. Ford-Fulkerson terminates in at most nC iterations.

Running time. $O(mnC)$

Exponential in input size:
$\log C$ bits (to represent C)

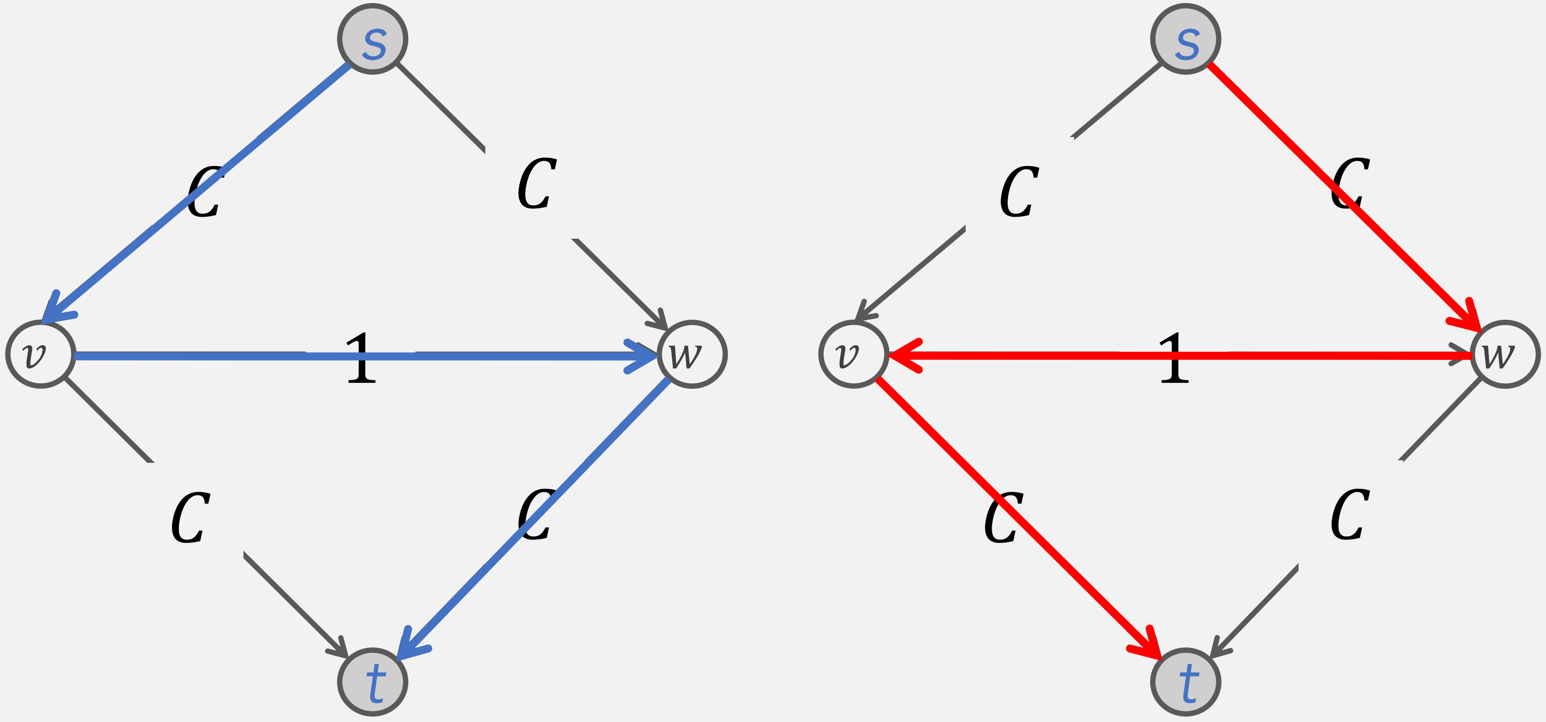
Can it be this bad?

Ford-Fulkerson: exponential example

Obs. If max capacity is C, then FF can take $\geq C$ iterations.

- $s \to v \to w \to t$
- $s \to w \to v \to t$
- $s \to v \to w \to t$
- $s \to w \to v \to t$
- $\ldots$
- $s \to v \to w \to t$
- $s \to w \to v \to t$

Each augmenting path
sends only 1 unit of flow
(# augmenting paths = $2C$)



Choosing good augmenting paths

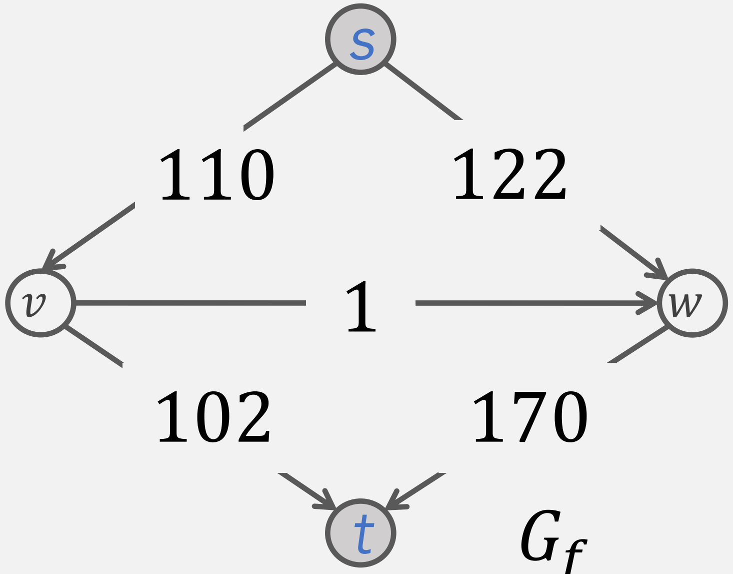
- Use care when selecting augmenting paths
  - Some choices lead to exponential algorithms
  - Clever choices lead to polynomial algorithms
  - If capacities are irrational, algorithm not guaranteed to terminate!

- Good choices of augmenting paths [EdmondsKarp'72,Dinitz'70]
  - Max bottleneck capacity [Next]
  - Fewest edges (shortest) [CLRS 26.2]

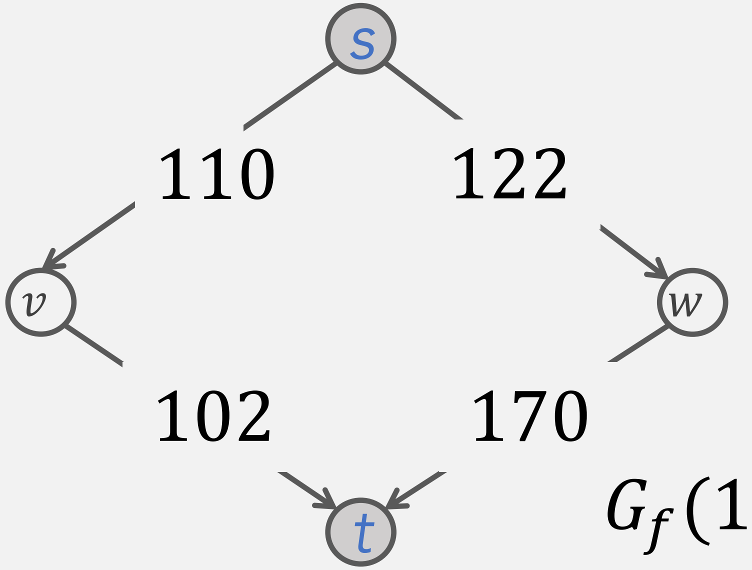
Capacity scaling

Intuition. Choosing path with highest bottleneck capacity increases the flow by max possible amount

- OK to choose sufficiently large bottleneck: scaling parameter Δ
- Let $G_f(\Delta)$ be the subgraph of the residual graph consisting of only arcs with capacity at least Δ



G_f



$G_f(100)$

Capacity scaling algorithm

Scaling−Max−Flow (G, s, t, c)
For each $e \in E$ $f(e) \leftarrow 0$,
$G_f \leftarrow residual\ graph$
$\Delta \leftarrow$ smallest power of 2 & $\geq C$

While $\Delta \geq 1$
  $G_f(\Delta) \leftarrow \Delta$−residual graph
  While there is an augmenting path P in $G_f(\Delta)$
    $f \leftarrow Augment(f, c, P)$ // augment flow by $\geq \Delta$
    Update $G_f(\Delta)$
  $\Delta \leftarrow \Delta/2$
return f

Exercise. Prove correctness

Capacity scaling algorithm: running time

While $\Delta \geq 1$
  $G_f(\Delta) \leftarrow \Delta-$residual graph
  While there is P in $G_f(\Delta)$
    $f \leftarrow Augment(f, c, P)$
    Update $G_f(\Delta)$
  $\Delta \leftarrow \Delta/2$
…

Lemma1 Outer loop runs $1 + \log C$ times.

Pf. Initially $C \leq \Delta \leq 2C$, decreases by a factor of 2 each iteration

Lemma2. Let f be the flow at the end of a Δ-scaling phase. Then the value of the maximum flow f^* is at most $v(f) + m\Delta$.

Lemma3. There are at most $2m$ augmentations per scaling phase.

Pf. Let f be the flow at end of previous scaling

- [Lemma2] $\Rightarrow v(f^*) \leq v(f) + m(2\Delta)$
- Each augmentation in Δ-scaling increases f by Δ

Theorem. Scaling-max-flow finds a max flow in $O(m^2 \log C)$ time.

Completing the proof

Lemma2. Let f be the flow at the end of a Δ-scaling phase. Then the value of the maximum flow f^* is at most $v(f) + m\Delta$.

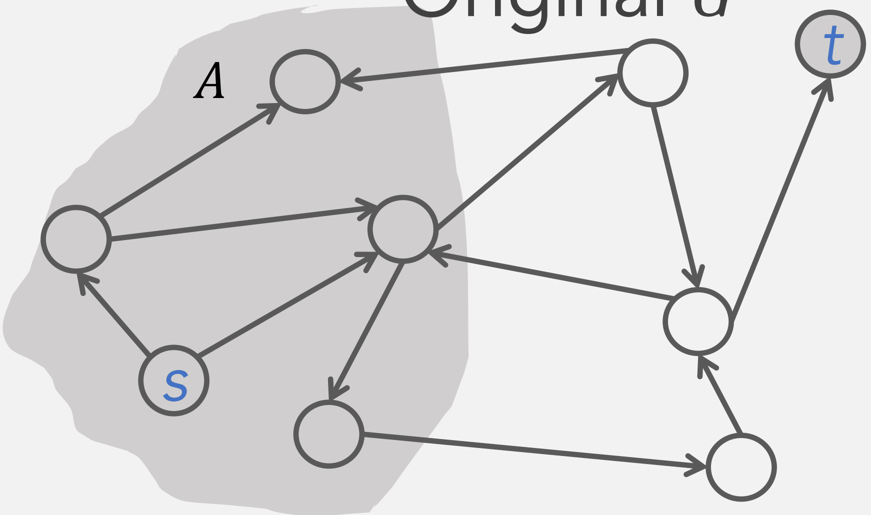
Pf. [Almost identical to proof of max-flow min-cut theorem]

Show cut (A, B) w. $cap(A, B) \le v(f) + m\Delta$ at the end of a $\Delta-$phase.

- Choose A to be the set of nodes reachable from s in $G_f(\Delta)$
- By definition $s \in A$ & $t \notin A$

$$\begin{aligned} v(f) &= \textstyle\sum_{e \text{ outof } A} f(e) - \sum_{e \text{ into } A} f(e) \\ &\ge \textstyle\sum_{e \text{ outof } A} (c(e) - \Delta) - \sum_{e \text{ into } A} \Delta \\ &= \textstyle\sum_{e \text{ outof } A} c(e) - \sum_{e \text{ outof } A} \Delta - \sum_{e \text{ into } A} \Delta \\ &= cap(A, B) - m\Delta \end{aligned}$$

Original G



Augmenting-path algorithms: summary

Year	Method	# augmentations	Running time
1955	Augmenting path	nC	$O(mnC)$
1972	Fattest path	$m \log mC$	$O(m^2 \log n \log mC)$
1972	Capacity scaling	$m \log C$	$O(m^2 \log C)$
1985	Improved CapS	$m \log C$	$O(mn \log C)$
1970	Shortest path	mn	$O(m^2 n)$
1970	level graph	mn	$O(mn^2)$
1983	dynamic trees	mn	$O(mn \log n)$

and the show goes on …

Year	Method	Worst case	Discovered by
1951	Simplex	$O(mn^2C)$	Dantzig
1955	Augmenting path	$O(mn^2W)$	Ford-Fulkerson
…			
1988	Push-relabel	$O(mn\log(n^2/m))$	Goldberg-Tarjan
…			
2013	Compact networks	$O(mn)$	Orlin
2016	Electrical flows	$\tilde{O}(m^{10/7}C^{1/7})$	Madry
20XX			

To keep it simple, cite below when you invoke a max-flow subroutine in hw/exam

Maximum flows can be computed in $O(mn)$ time

Another formulation of max-flow problem

Recall. An $s\text{–}t$ flow is a function $f: E \to \mathbb{R}$ satisfying

- [Capacity] $\forall e \in E: 0 \le f(e) \le c(e)$
- [Conservation] $\forall v \in V \setminus \{s,t\}$: $\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$

The value of a flow f is $v(f) \coloneqq \sum_{e \text{ out of } s} f(e)$

Max–Flow Problem

Real-value variables $\vec{f} = \{f_e : e \in E\}$

Maximize: $v(\vec{f})$

Subject to:

$$0 \le f_e \le c(e), \ \forall e \in E$$

$$\sum_{e \text{ into } v} f_e - \sum_{e \text{ out of } v} f_e = 0, \ \forall v \in V \setminus \{s,t\}$$

Linear constraints: no $x^2, xy, \sin(x)$, ...

Grade maximization

Input. HW from two courses (xxx & 629) due in one day

- Every hour you spend, you earn 1pts on xxx or 5pts on 629
- Your brain will explode if you work more than 12hrs on xxx or 15hrs on 629
- Of course, there are only 24 hrs in a day

Goal. Maximize the total pts you can earn

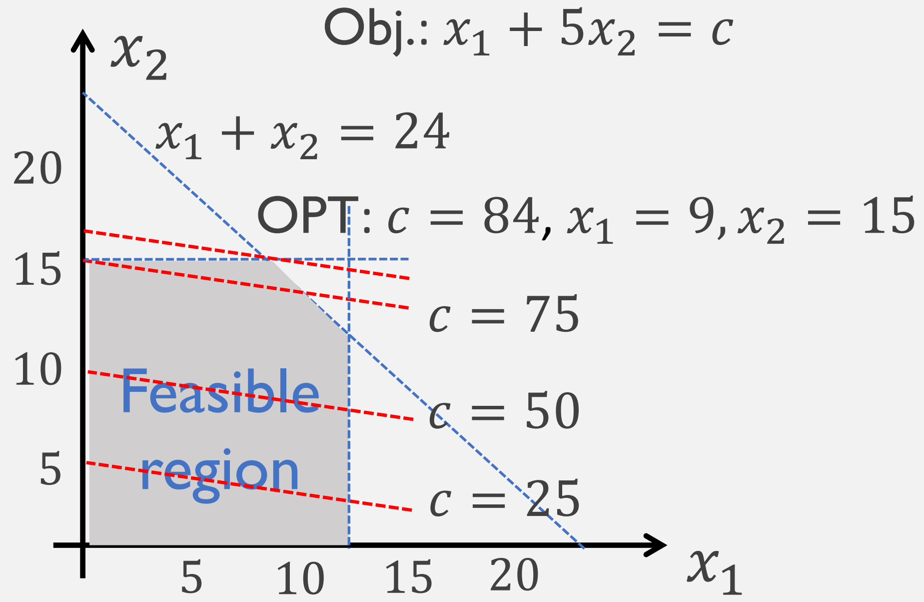
Grade—Maximization

Variables: x_1 (xxx hrs); x_2 (629 hrs)

Maximize: $x_1 + 5x_2$

Subject to: // linear constraints

$$0 \le x_1 \le 12$$
$$0 \le x_2 \le 15$$
$$x_1 + x_2 \le 24$$



Linear programming

Linear programming. Optimize a linear objective function subject to linear inequalities.

- Formal definition and representations
- Duality
- Algorithms: simplex, ellipsoid, interior point

Why significant?

- Design poly-time algorithms & approximation algorithms
- Wide applications: math, economics, business, transportation, energy, telecommunications, and manufacturing

Ranked among most important scientific advances of 20th century

Happy Halloween!
&
Enjoy the treats!

Ozzy - Mr Crowley	https://www.youtube.com/watch?v=vDVLMS_Yhe4
葬尸湖 – 弈秋	https://www.youtube.com/watch?v=Hb2OgPoZGIY
Uaral - Lament	https://www.youtube.com/watch?v=azZz8Oyz-O0
Saturnus - All alone	https://www.youtube.com/watch?v=-FID_X6hfmc
Forest of Shadows - Eternal Autumn	https://www.youtube.com/watch?v=usGGX6ZqQv0