F, 10/25/19

Fall'19 CSCE 629

Analysis of Algorithms

Guest lecture by
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Lecture 22

- Bipartite matching

Credit: based on slides by A. Smith & K. Wayne

Recap: max-flow & min-cut

- Ford-Fulkerson. Find max s-t flow & min s-t cut in time $O(mnC)$
  - All capacities are integers $\leq C$
  - Will see how to remove this

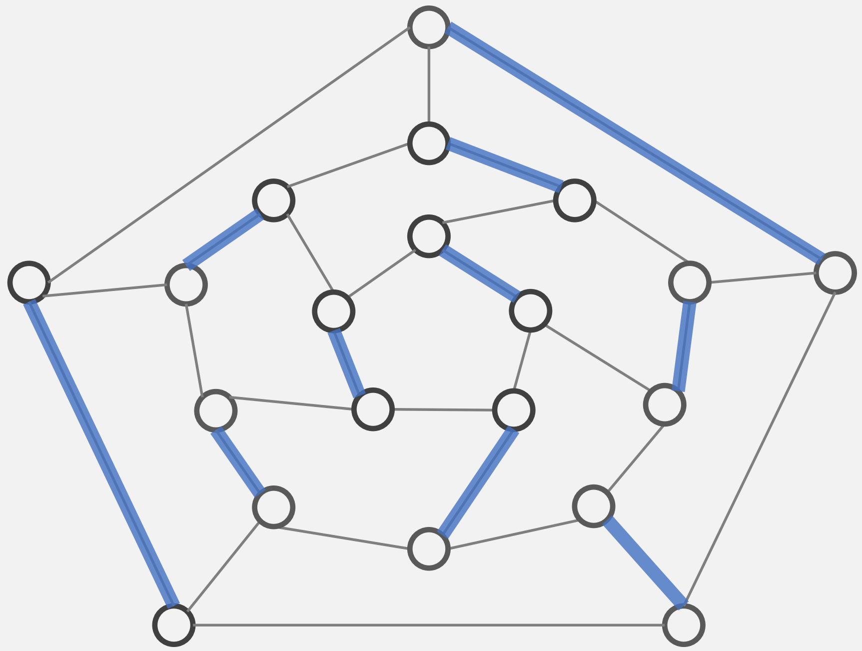
- Duality. max flow value = min cut capacity

- Integrality theorem. If all capacities are integers, then there exists a max flow f for which every flow value $f(e)$ is an integer.

- Today. Applications when $C = 1$ [N.B. running time $O(mn)$]

Matching

Def. Given an undirected graph $G = (V, E)$. A subset of edges $M \subseteq E$ is a matching if each node appears in at most one edge in M.



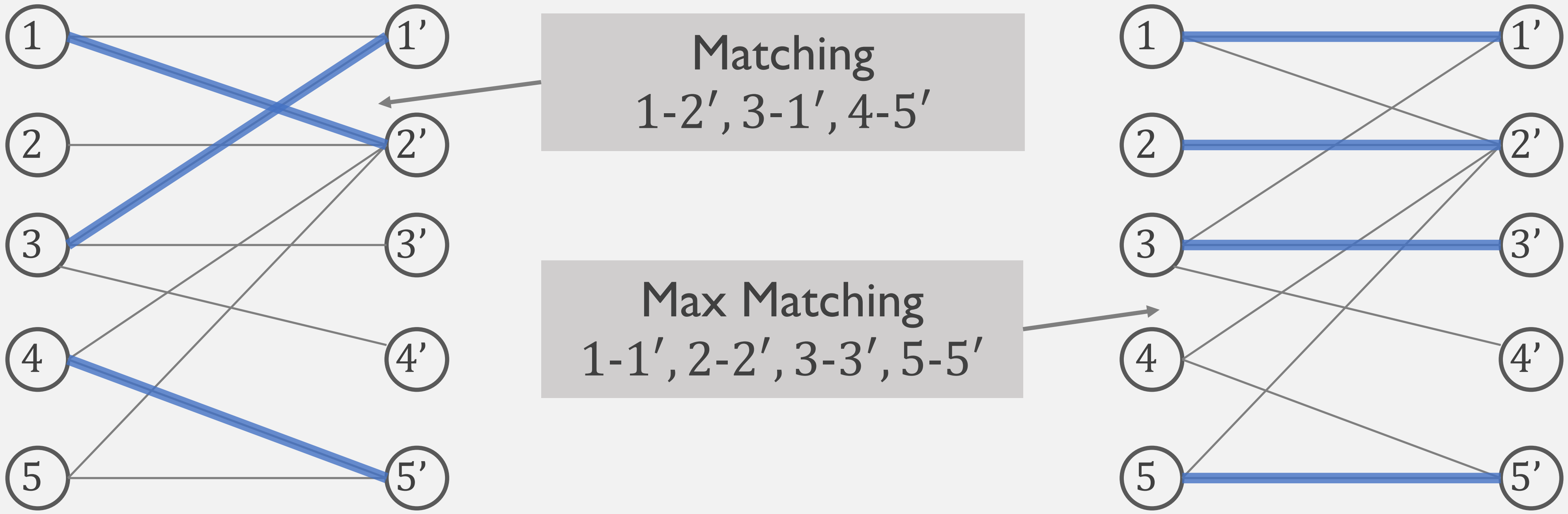
Max matching. Find a matching of max cardinality.

- i.e., adding any edge will make it no longer a matching

Bipartite Matching

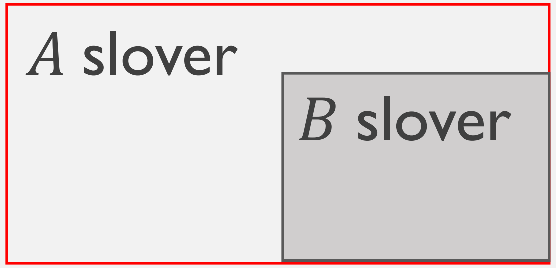
Bipartite graph. A graph G is bipartite if the nodes V can be partitioned into two subsets L and R such that every edge connects a node in L with a node in R.

(Max) Bipartite matching. Given a bipartite graph $G = (L \cup R, E)$, find a max cardinality matching.

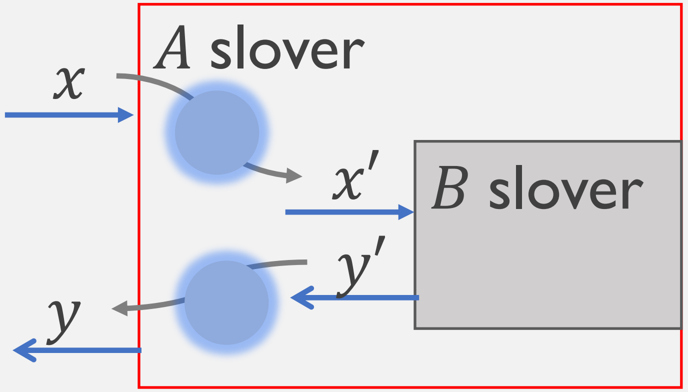


Reductions

- Informal. Problem A reduces to problem B if there is a simple algorithm for A that uses an algorithm for B as a subroutine.



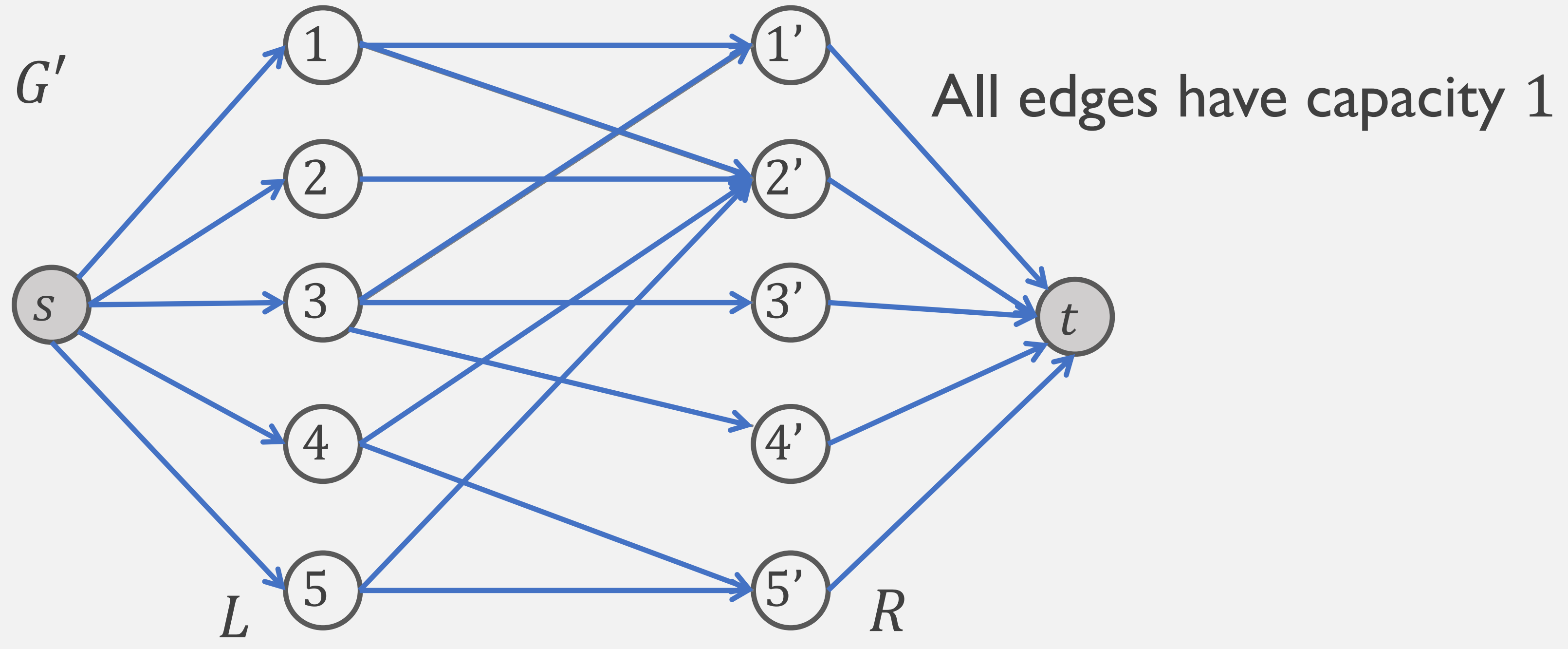
- Common scenario [a.k.a. Karp reduction]
  - Given instance x of problem A.
  - Convert x to an instance x' and solve it.
  - Use the solution to x' to build a solution for x.



- Useful skill
  - Quickly identifying problems where existing solutions may be applied
  - Good programmers do this all the time [don't reinvent wheels]

Reducing bipartite matching to max flow

- Reduction to max flow
  - Create directed graph $G' = (L \cup R \cup \{s, t\}, E')$
  - Direct all edges from L to R, and assign capacity 1
  - Add source s, and capacity 1 edges to every node in L
  - Add sink t, and capacity 1 edges from each node in R to t.

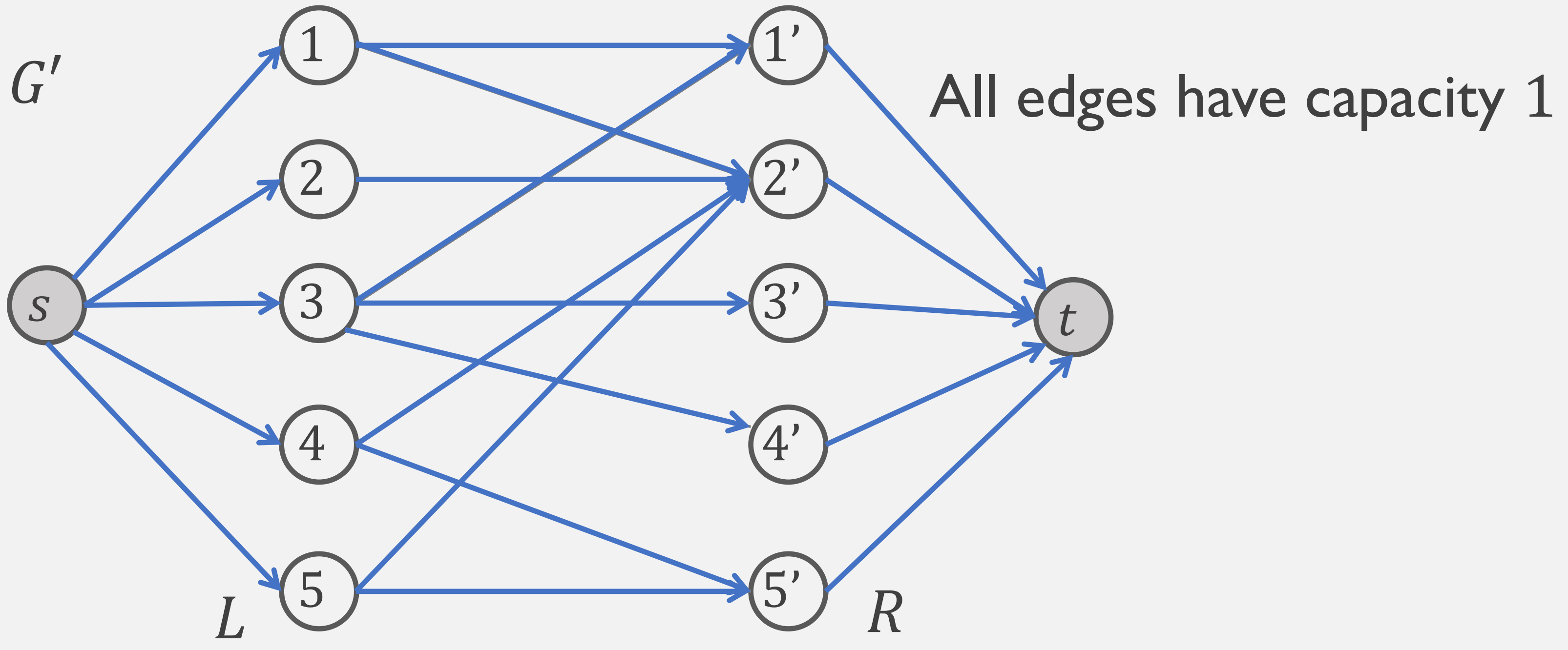


Bipartite matching: proof of correctness

Theorem. Max cardinality matching in G = value of max flow in G'

Proof. We show two claims

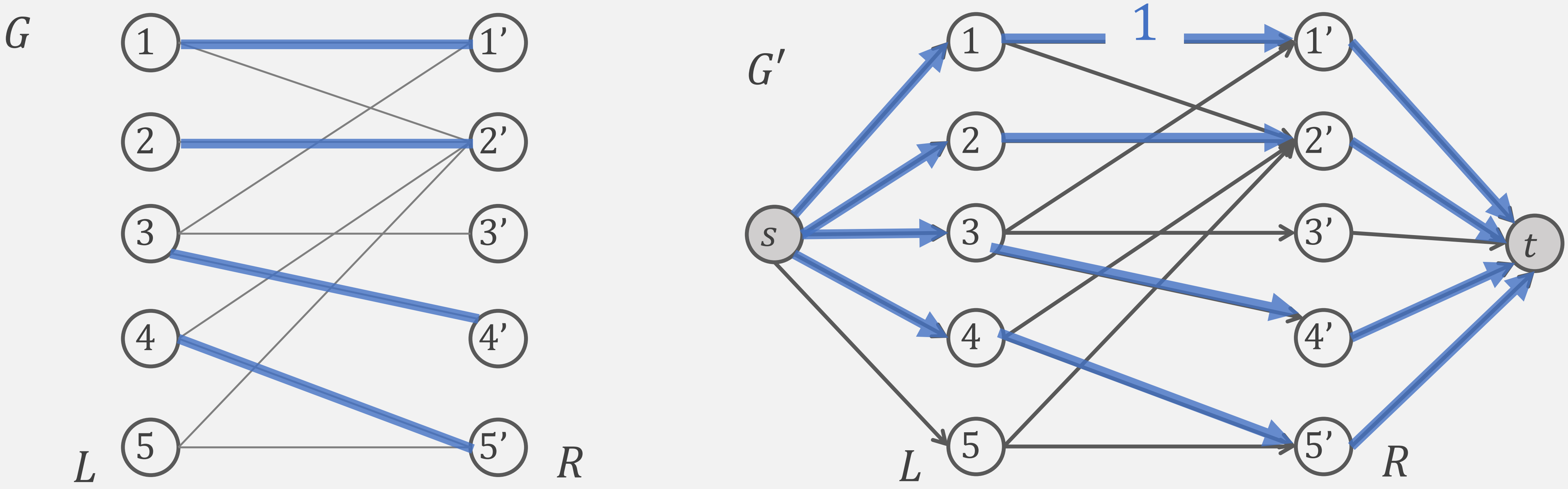
- Max matching in $G \leq$ max flow in G'
- Max matching in $G \geq$ max flow in G'



Bipartite matching: proof of correctness

Proof. (Part 1) Max matching in $G \leq$ max flow in G'

- Given max matching M of cardinality k
- Consider flow f that send 1 unit along each of k paths
- f is a flow of value k



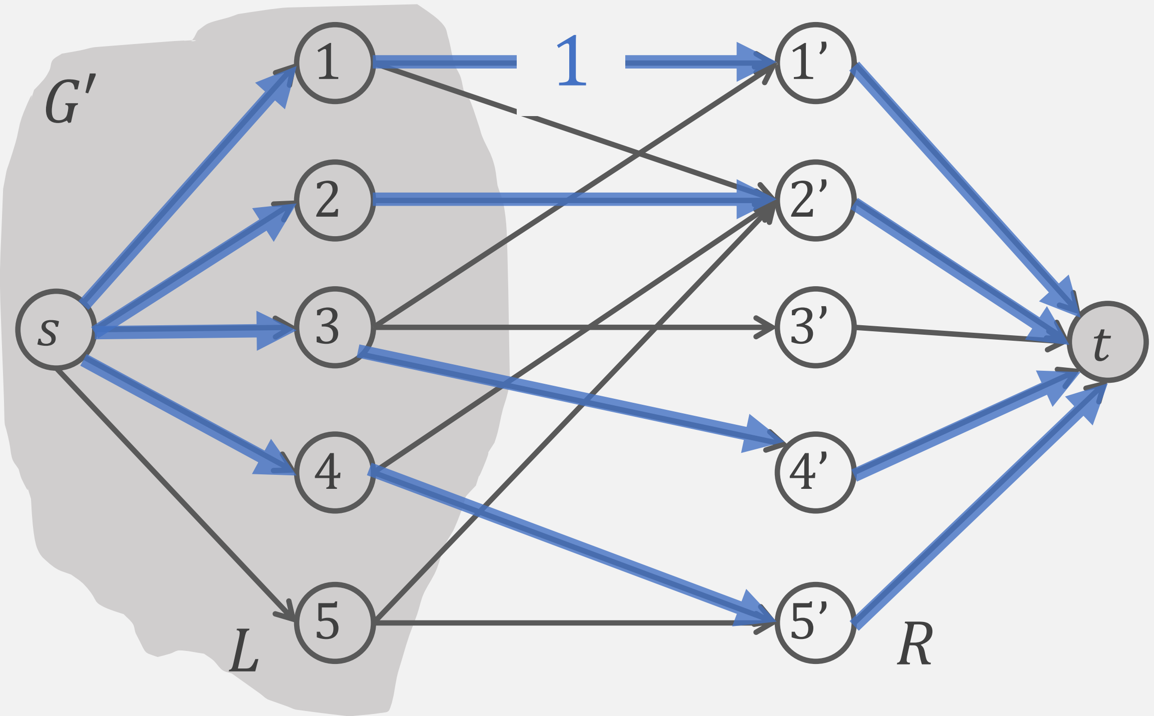
Bipartite matching: proof of correctness

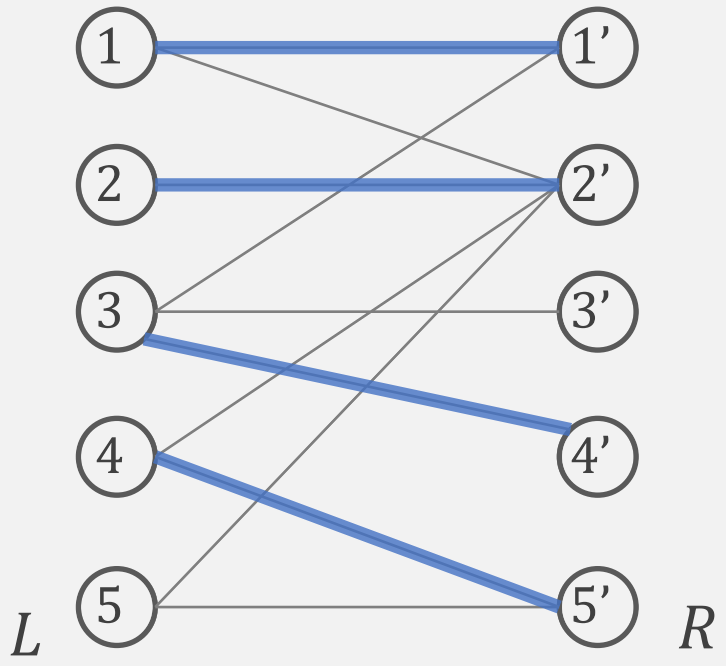
Proof. (Part 2) Max matching in $G \geq$ max flow in G'

- Given max flow f in G' of integer value k [Exists by Integrality theorem]
- All capacities are $1 \Rightarrow f(\mathrm{e})$ is 0 or 1. Let $M =$ edges from L to R with $f(e) = 1$

$\Rightarrow M$ is a matching (each node in L and R participate in at most one edge)

& M has size k (consider cut $(s \cup L, R \cup t)$).





Perfect matching

Def. A matching $M \subseteq E$ is perfect if each node appears in exactly one edge in M.

When does a bipartite graph have a perfect matching?

Structure of bipartite graphs with perfect matchings.

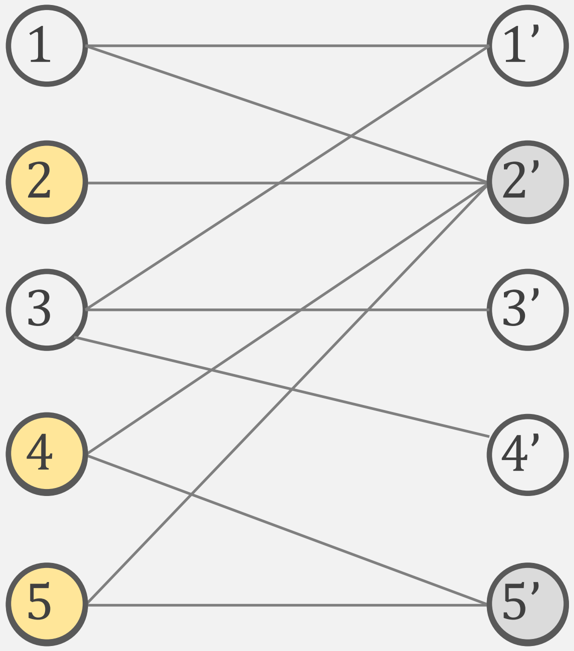
- Clearly we must have $|L| = |R|$
- What other conditions are necessary?
- What conditions are sufficient?

Perfect matching

Notation. Let S be a subset of nodes. Let $N(S)$ be the set of nodes adjacent to nodes in S.

Observation. If a bipartite graph $G = (L \cup R, E)$, has a perfect matching, then $|N(S)| \geq |S|$ for all subsets $S \subseteq L$.

- Pf. Each node in S has to be matched to a different node in $N(S)$.



No perfect matching
$S = \{2,4,5\}, N(S) = \{2', 5'\}$

Marriage theorem

Actually, this is also a sufficient condition …

Marriage Theorem [Frobenius1917, Hall1935]

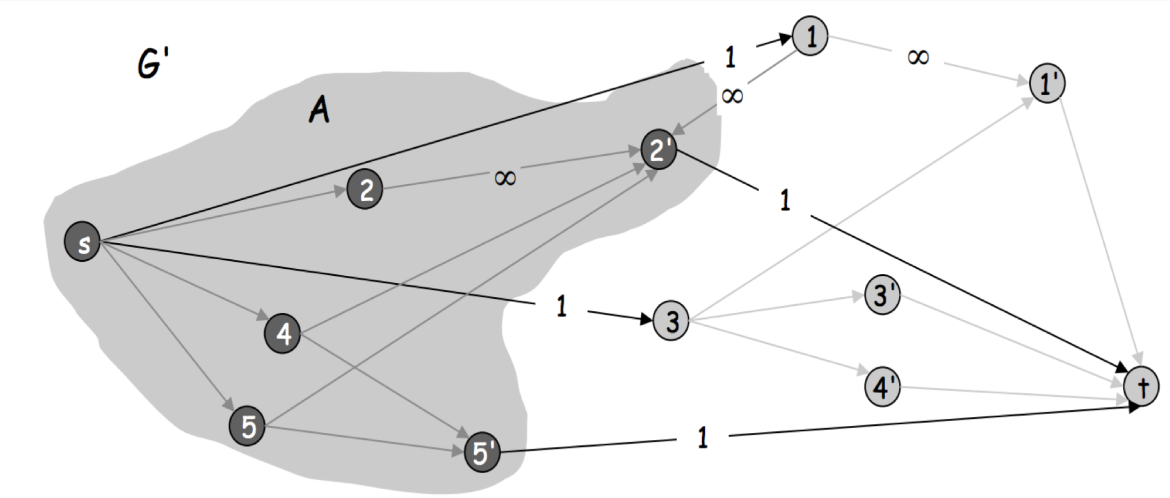
Let $G = (L \cup R, E)$ be a bipartite graph with $|L| = |R|$. Then G has a perfect matching if and only if $|N(S)| \geq |S|$ for all subsets $S \subseteq L$.

Pf. ⇒ (“only if”) This is the previous observation.

Marriage theorem: proof

Pf. $\Leftarrow$ ("if") Suppose for contradiction $|N(S)| \geq |S|$ for all subsets $S \subseteq L$, but G does not contain a perfect matching.

- Formulate as a max flow problem in G' with ∞ capacities on edges from L to R.
- Let (A, B) be min cut in G'. By max-flow min-cut, $cap(A, B) < |L|$
- Let $L_A = L \cap A, L_B = L \cap B, R_A = R \cap A$. Then $cap(A, B) = |L_B| + |R_A|$.
  $\Rightarrow |R_A| = cap(A, B) - |L_B| < |L| - |L_B| = |L_A|$
- Since min cut cannot use ∞ edges, $N(L_A) \subseteq R_A$. $|N(L_A)| \leq |R_A| < |L_A|$!!!



$$L = \{1 \ldots 5\}, R = \{1' \ldots 5'\}$$
$$L_A = \{2,4,5\}, L_B = \{1,3\}, R_A = \{2', 5'\}$$
$$N(L_A) = \{2', 5'\}$$