W, 10/23/19

Fall'19 CSCE 629

Analysis of Algorithms

Guest lecture by
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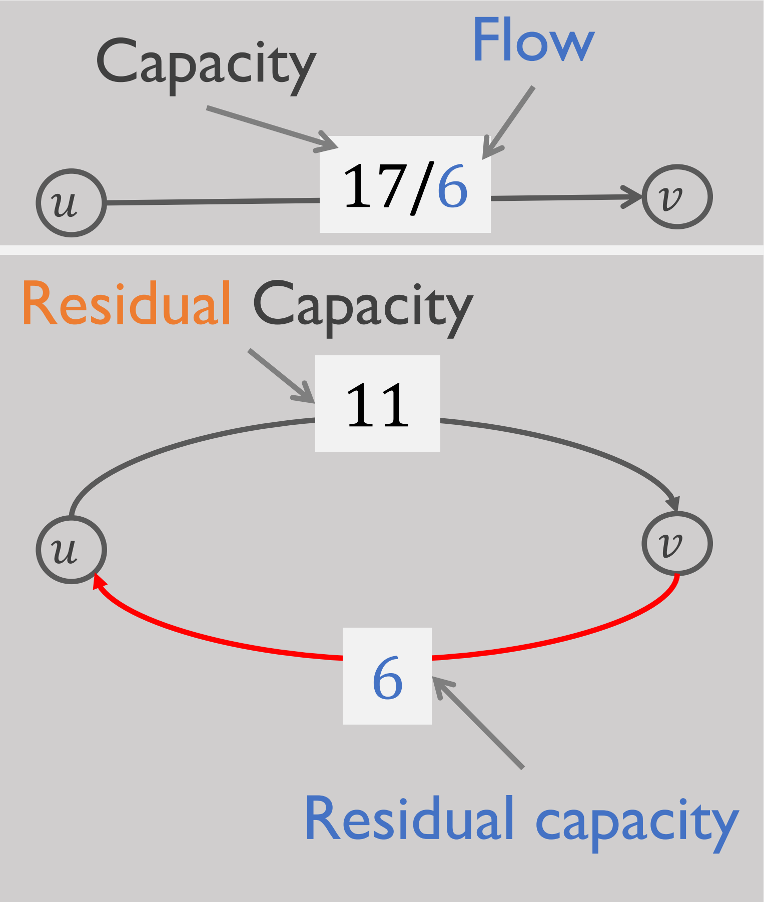
Lecture 21

- Ford-Fulkerson algorithm

Credit: based on slides by A. Smith & K. Wayne

Recap: residue graph

- **Original edge:** $e = (u, v) \in E$
  - Flow $f(e)$, capacity $c(e)$
- **Residual edge:** "Undo" flow sent
  - $e = (u, v)$ and $e^R = (v, u)$
  - Residual capacity $$c_f(e) = \begin{cases} c(e) - f(e) & if\ e \in E \\ f(e) & if\ e^R \in E \end{cases}$$
- **Residual graph** $G_{f,c} = (V, E_{f,c})$
  - Residual edges with positive residual capacity
  - $E_{f,c} = \{e: f(e) < c(e)\} \cup \{e^R: f(e) > 0\}$



- **Augmenting path.** Simple path $s \rightsquigarrow t$ in residual graph G_f.
  - Bottleneck capacity of an augmenting path P: minimum residual capacity of any edge in P.

Recap: Augmenting path theorem

Theorem. f is a max flow iff. no augmenting paths ($s \leadsto t$) in G_f.

Proof. We show that the following are equivalent ($a \Rightarrow b \Rightarrow c \Rightarrow a$)

a. f is a max flow
b. There is no augmenting path (with respect to f)
c. There exists a cut (A, B) such that $cap(A, B) = v(f)$

a $\Leftrightarrow$ c

Max-flow min-cut theorem

Value of max flow = capacity of min cut

Recap: proof of augmenting path theorem

a. f is a max flow
b. There is no augmenting path (with respect to f)
c. There exists a cut (A, B) such that $cap(A, B) = v(f)$

- a ⇒ b. We show contrapositive ¬b ⇒ $\neg a$

Lemma 1 (Augmented flow). Let P be an augmenting path with respect to f. Then f' below is a feasible flow with $v(f') > v(f)$.

$\delta \leftarrow$ bottleneck capacity of augmenting path P

For each $e \in P$, $f'(e) = \begin{cases} f(e) + \delta, & e \in E \\ f(e) - \delta, & e^R \in E \end{cases}$

Pf.

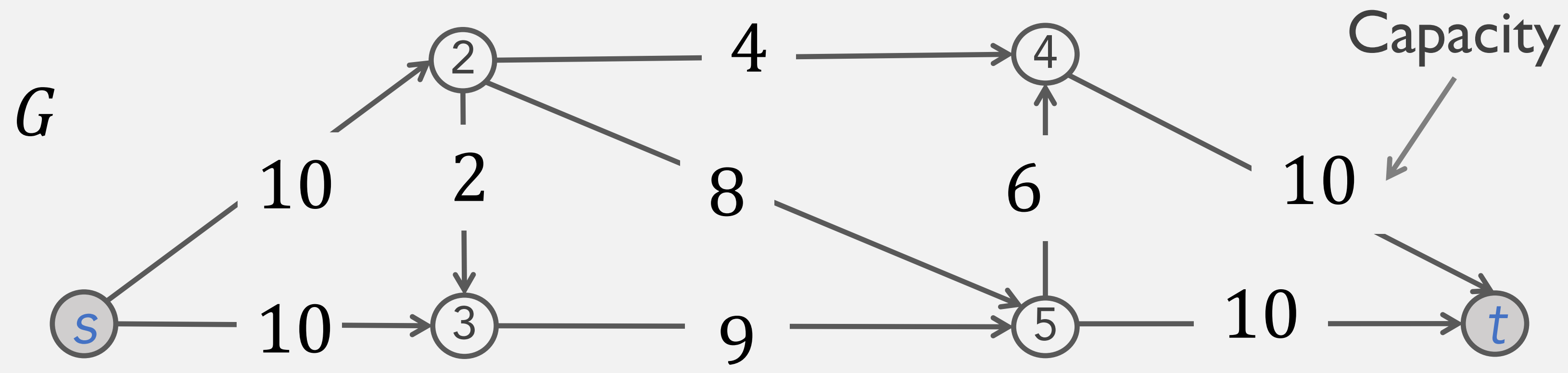
- Exercise. Verify f' is a feasible flow (i.e., capacity and conservation hold).
- $v(f') = v(f) + \delta > v(f)$ because only first edge in P leaves s.

Ford-Fulkerson augmenting-path algorithm

$\boldsymbol{Augment}(f, c, P)$
  $\delta \leftarrow$ bottleneck capacity of P
  For each $e \in P$
    If $e \in E$ $f'(e) = f(e) + \delta$
    Else $f'(e^R) = f(e^R) - \delta$
  return f'

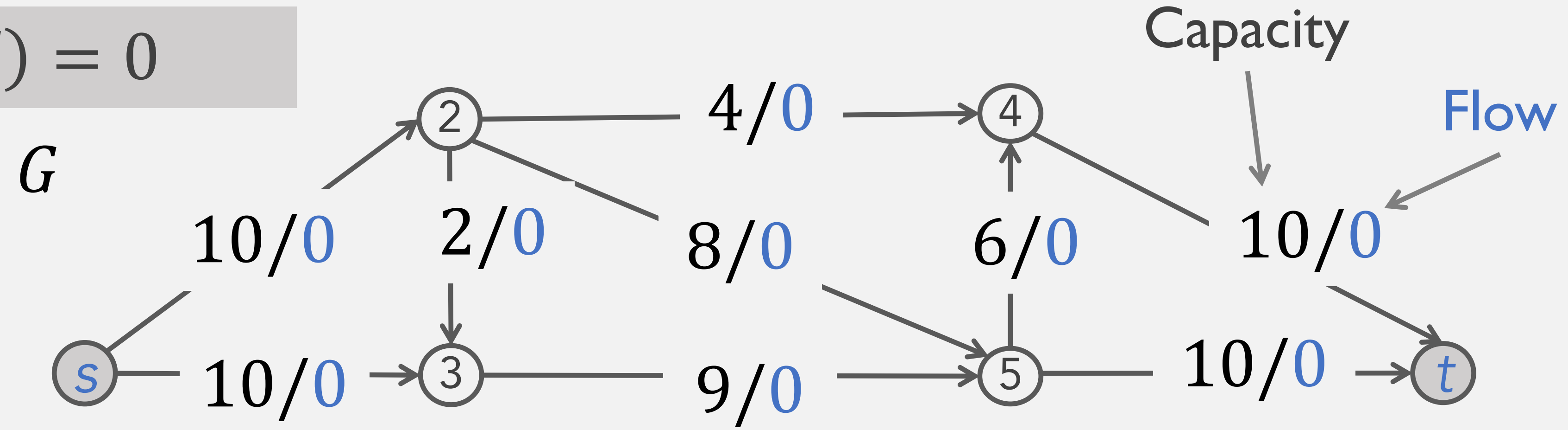
$\boldsymbol{Ford\text{–}Fulkerson}(G, s, t, c)$
  For each $e \in E$ $f(e) \leftarrow 0$, $G_f \leftarrow residual\ graph$
  While there is an augmenting path P in G_f
    $f \leftarrow Augment(f, c, P)$
    Update G_f
  return f

Ford-Fulkerson Algorithm demo0

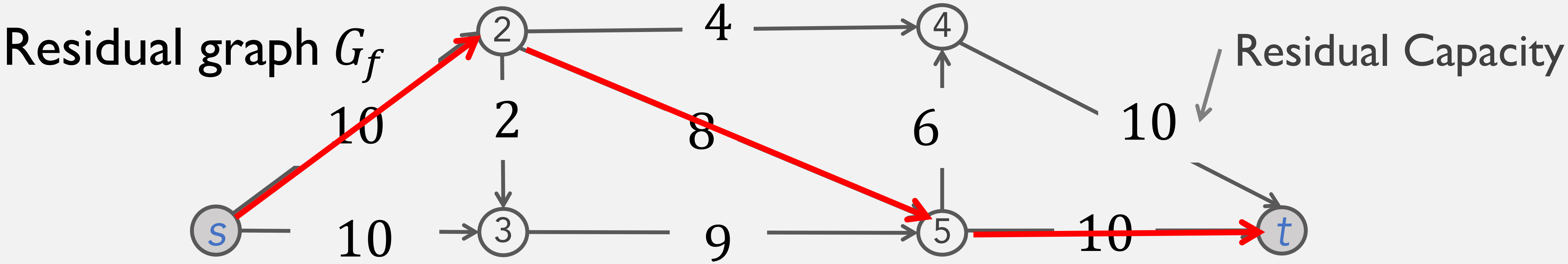


Ford-Fulkerson algorithm demo1

$v(f) = 0$

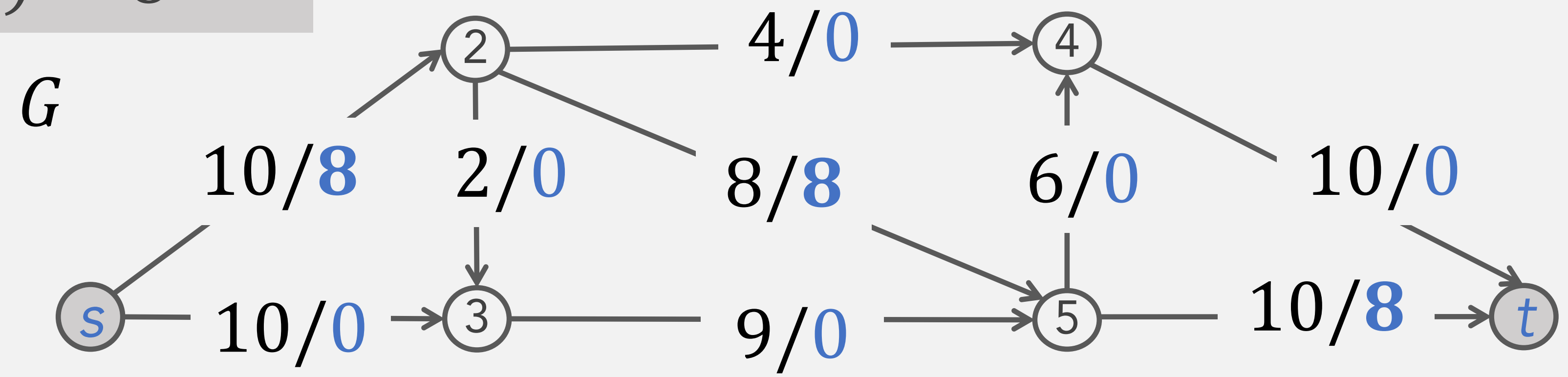


An augmenting path: $s - 2 - 5 - t, \delta = 8$

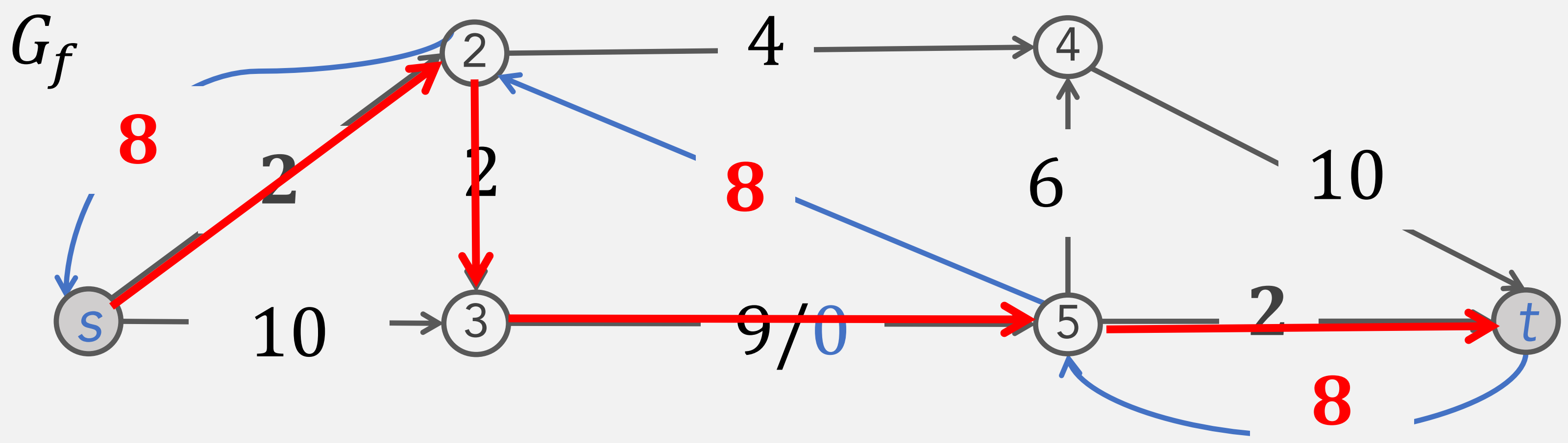


Ford-Fulkerson algorithm demo2

$v(f) = 8$

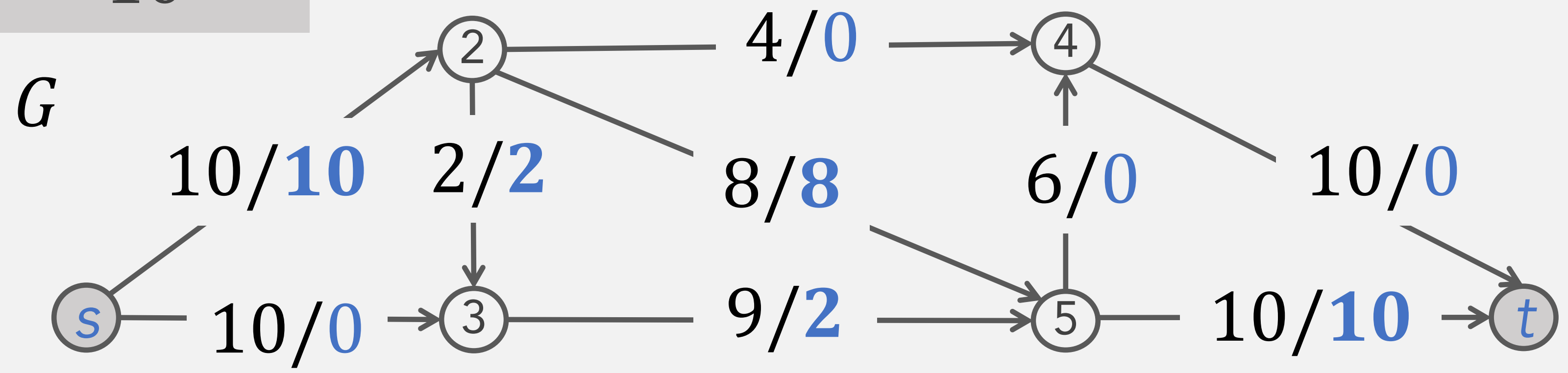


An augmenting path: $s - 2 - 3 - 5 - t, \delta = 2$

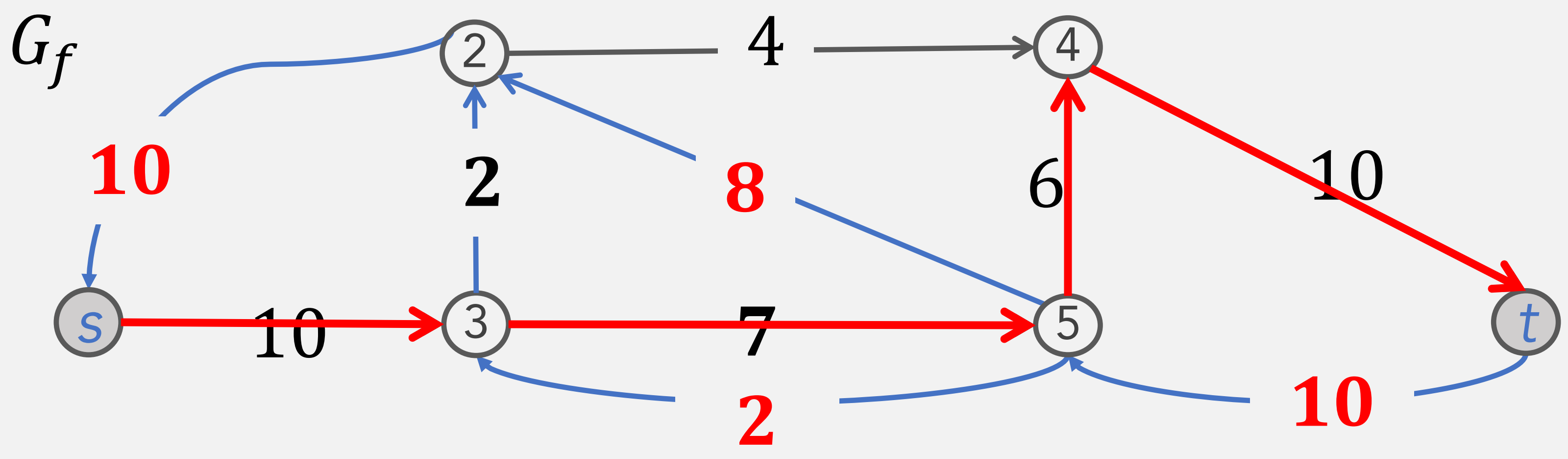


Ford-Fulkerson algorithm demo3

$v(f) = 10$

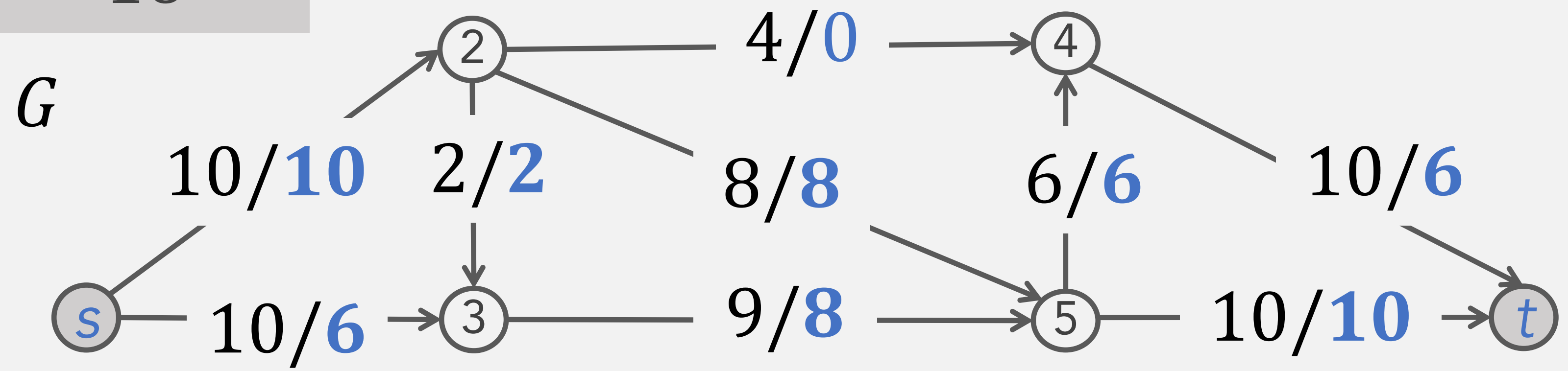


An augmenting path: $s - 3 - 5 - 4 - t, \delta = 6$

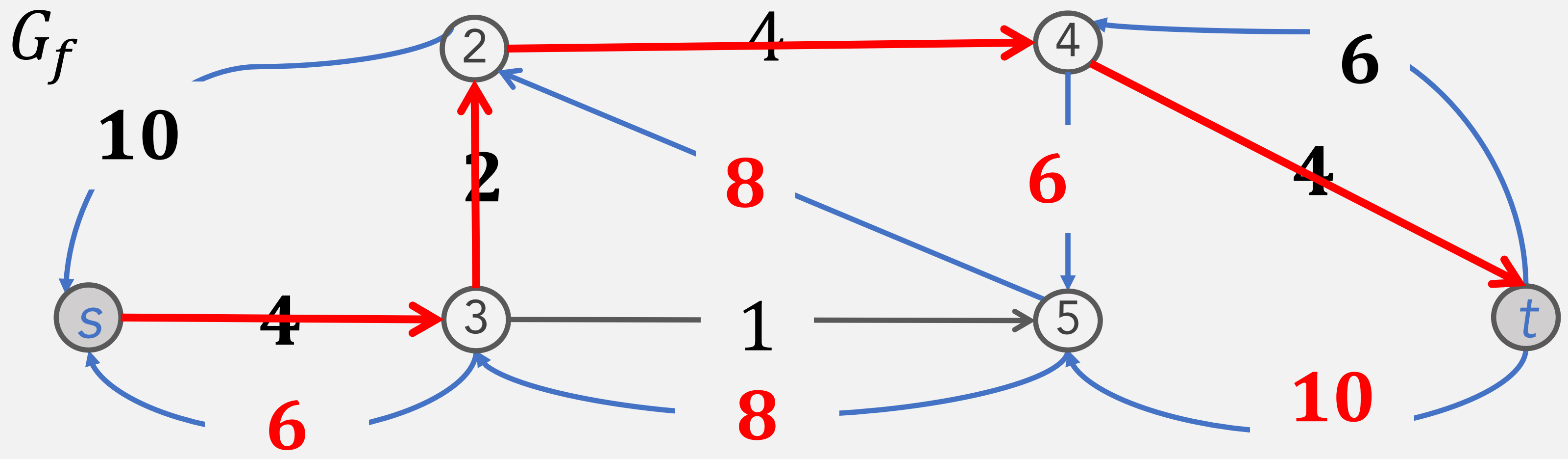


Ford-Fulkerson algorithm demo4

$v(f) = 16$

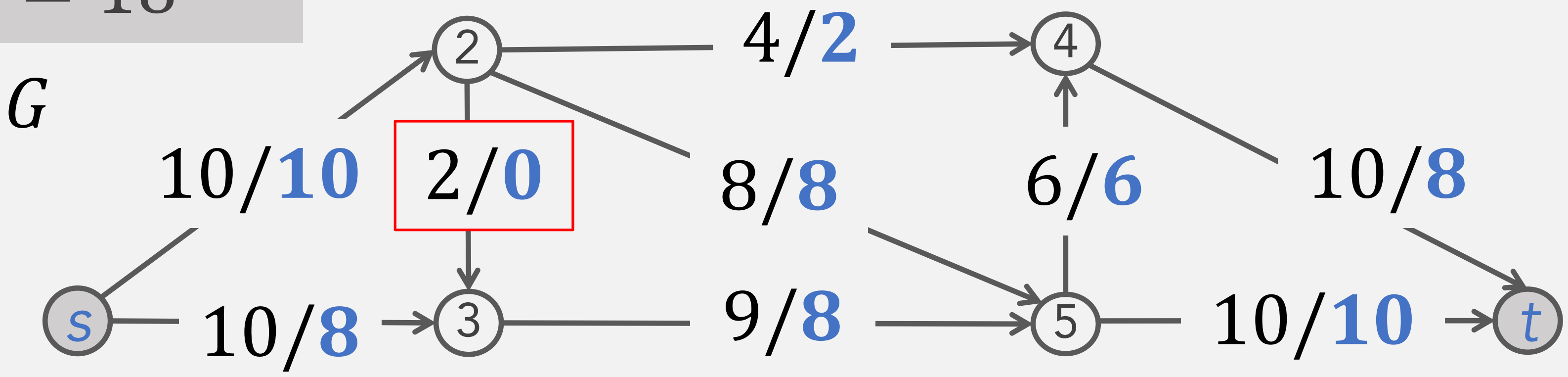


An augmenting path: $s - 3 - 2 - 4 - t, \delta = 2$

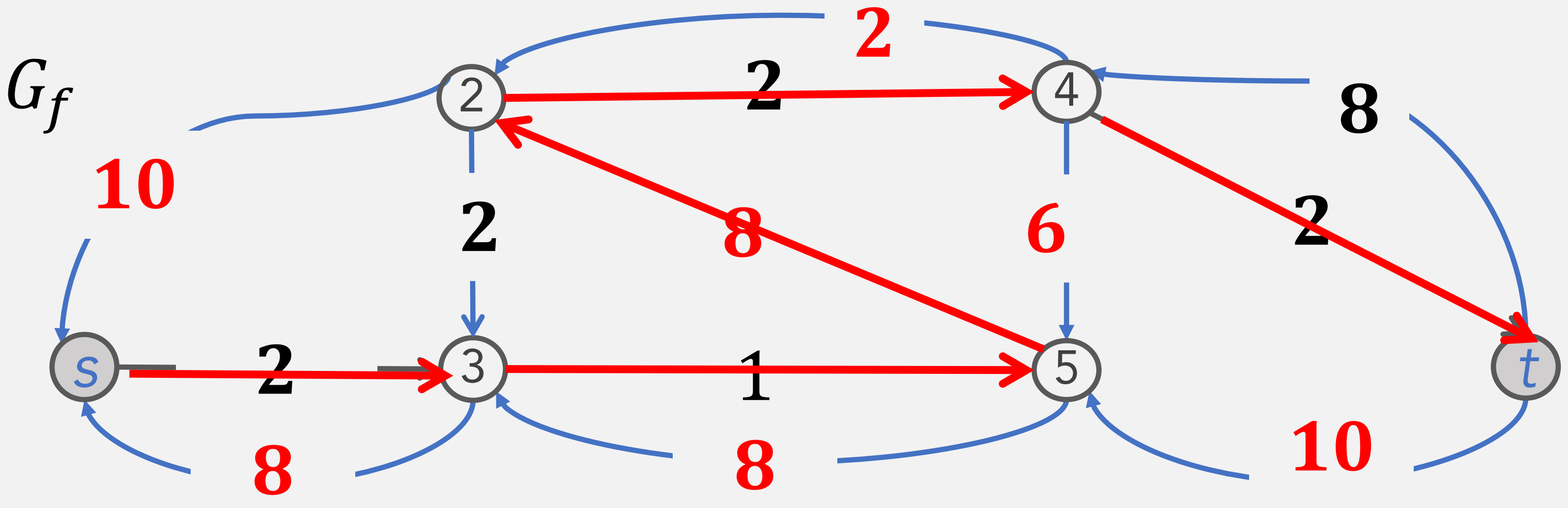


Ford-Fulkerson algorithm demo5

$v(f) = 18$

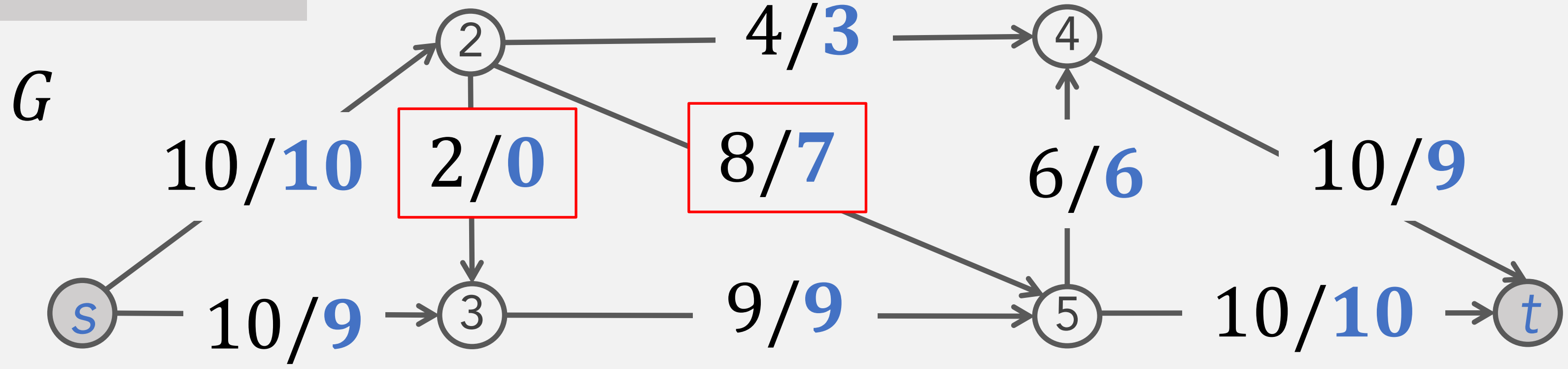


An augmenting path: $s - 3 - 5 - 2 - 4 - t, \delta = 1$

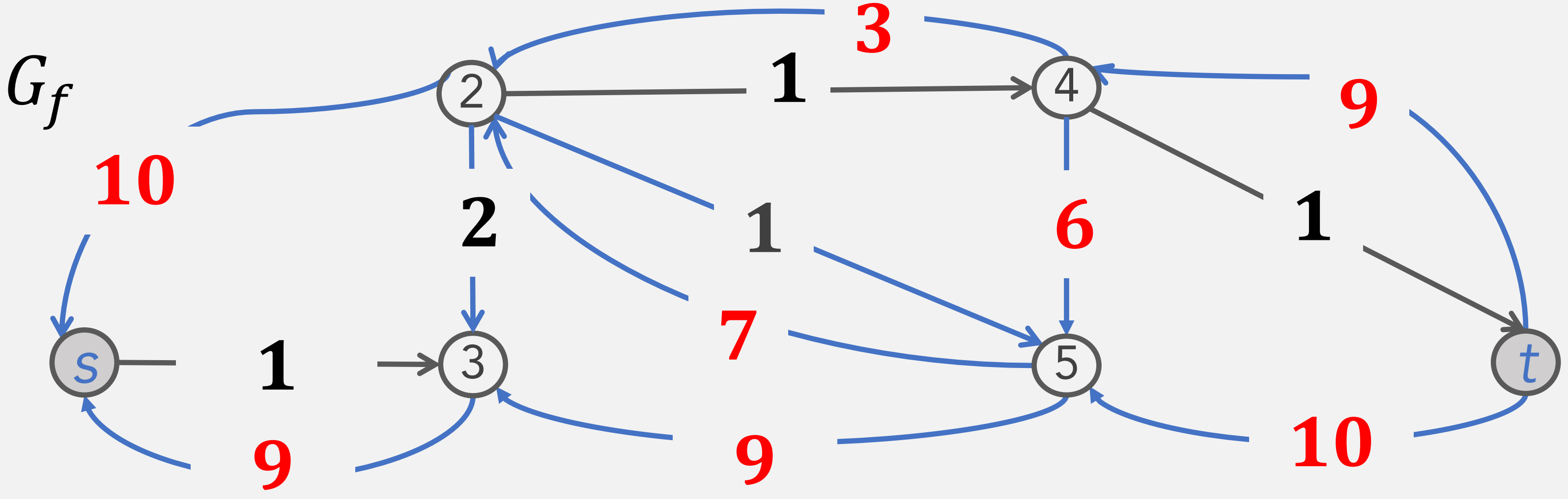


Ford-Fulkerson algorithm demo6

$v(f) = 19$



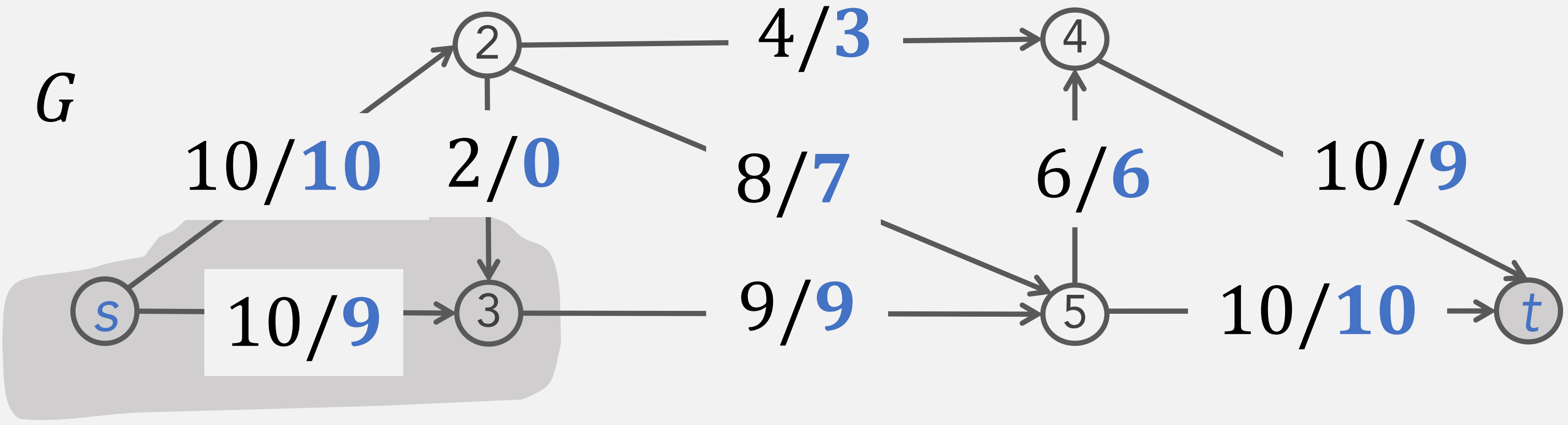
Any more augmenting paths?



Ford-Fulkerson algorithm demo7

$v(f) = 19$

Is this a max flow?



Cut $(A = \{s, 3\}, B = S \backslash A)$, $cap(A, B) = 19$

Ford-Fulkerson algorithm: summary so far

Ford–Fulkerson
While you can
- Greedily push flow
- Update residual graph

- Correctness. Augmenting path theorem.
- Running time. Does it terminate at all?

Ford-Fulkerson algorithm: analysis

- Assumption. All capacities are integers between 1 and C.
- Invariant. Every flow value $f(e)$ and every residual capacity $c_f(e)$ remains an integer throughout the algorithm.
- Theorem. Ford-Fulkerson terminates in at most nC iterations.

  Pf.
  - Each augmentation increases flow value by at least 1.
  - There are at most nC units of capacity leaving source s.

  Running time $O(mnC)$. Space $O(m+n)$

  Find an augmenting path in $O(m)$ time (by BFS/DFS).

- Integrality theorem. If all capacities are integers, then there exists a max flow f for which every flow value $f(e)$ is an integer.

More to come on further concerns/improvements …