M, 10/21/19

Fall'19 CSCE 629

Analysis of Algorithms

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Lecture 20

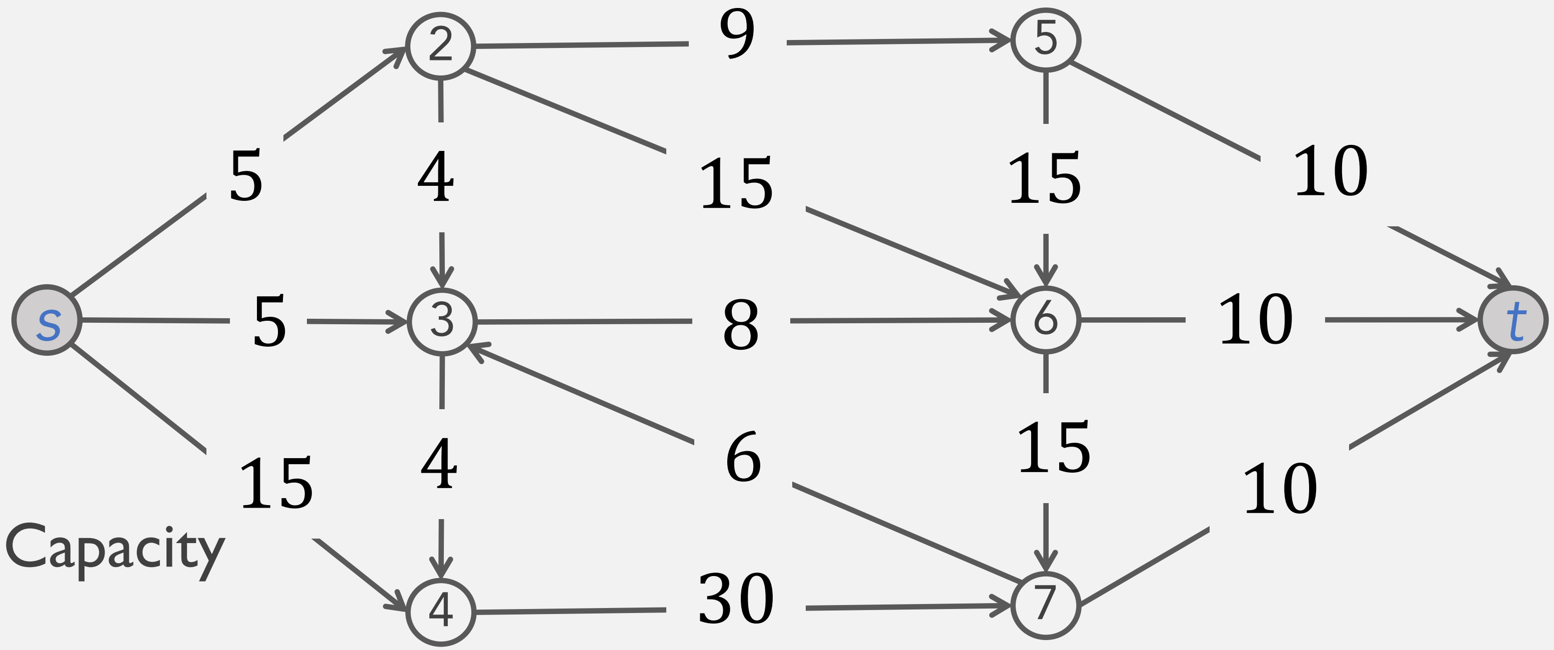
- Max-flow min-cut theorem

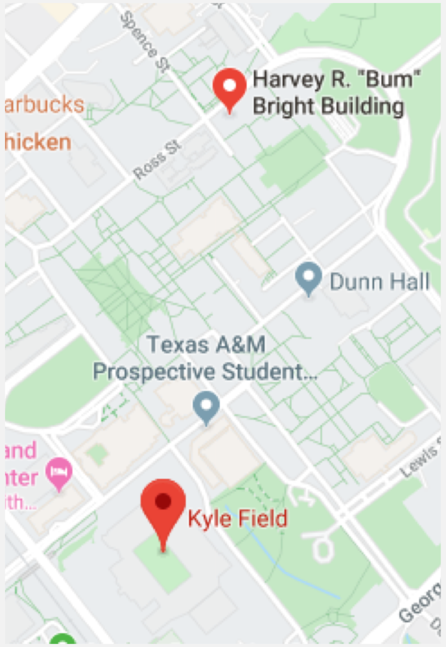
Credit: based on slides by A. Smith & K. Wayne

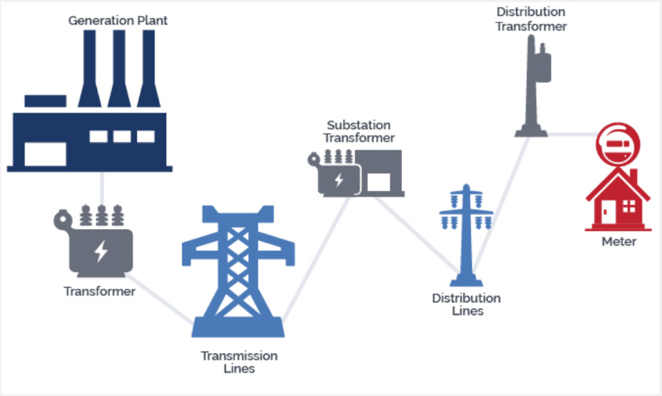
Recap: Flow network

Abstraction for material flowing through the edges

- $G = (V, E)$ directed graph, no parallel edges
- Two distinguished nodes: $s = source, t = sink$
- $\forall e \in E, c(e)$: capacity of edge e





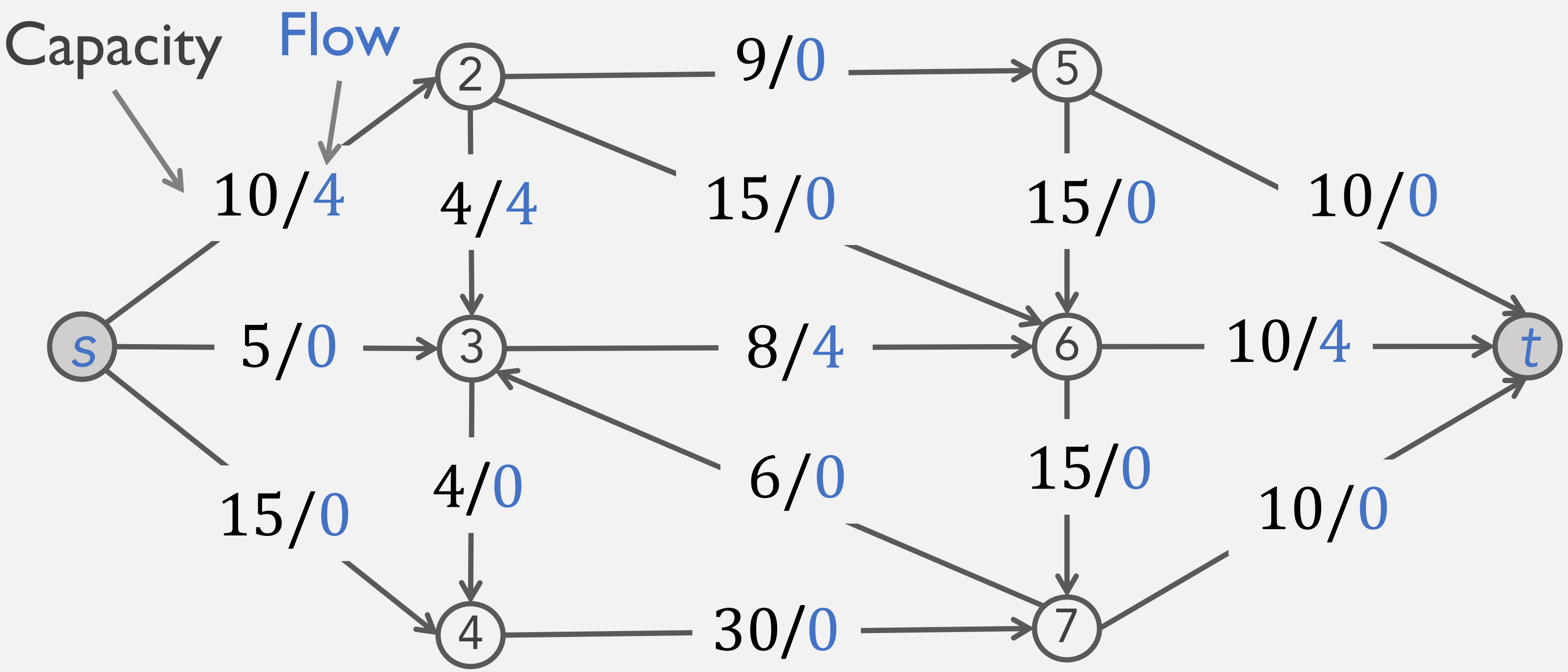


Flows

Def. An s–t flow is a function $f: E \to \mathbb{R}$ satisfying

- [Capacity] $\forall e \in E: 0 \leq f(e) \leq c(e)$
- [Conservation] $\forall v \in V \setminus \{s, t\}$: $\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$

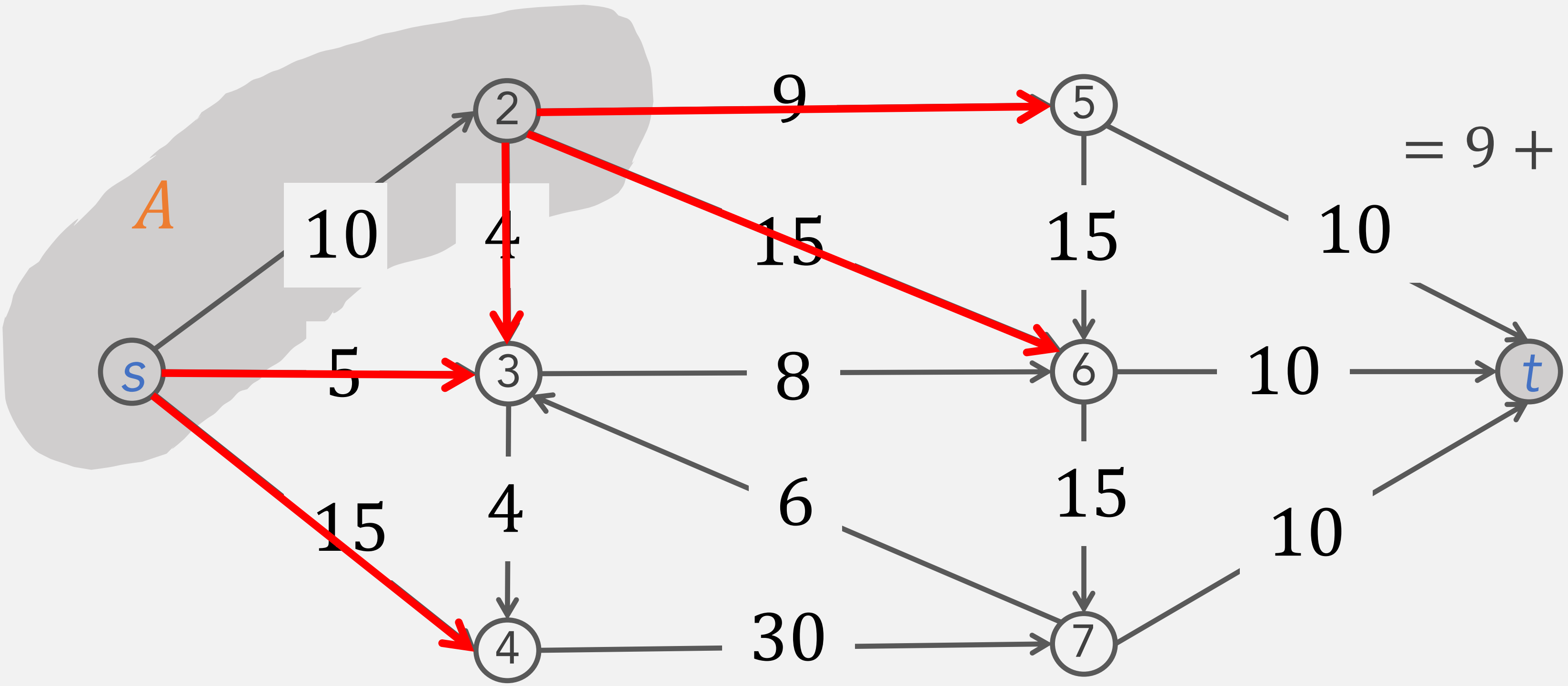
Def. The value of a flow f is $v(f) := \sum_{e \text{ out of } s} f(e)$



Value $v(f)$
$= 4 + 0 + 0 = 4$

Cuts

- Recall: a cut is a subset of nodes
- Def. s–t cut: $(A, B \coloneqq V \setminus A)$ partition of V with $s \in A$ & $t \in B$
- Def. Capacity of cut (A, B): $cap(A, B) = \sum_{e \text{ out of } A} c(e)$



$cap(A, B)$
$= 9 + 15 + 4 + 5 + 15 = 48$

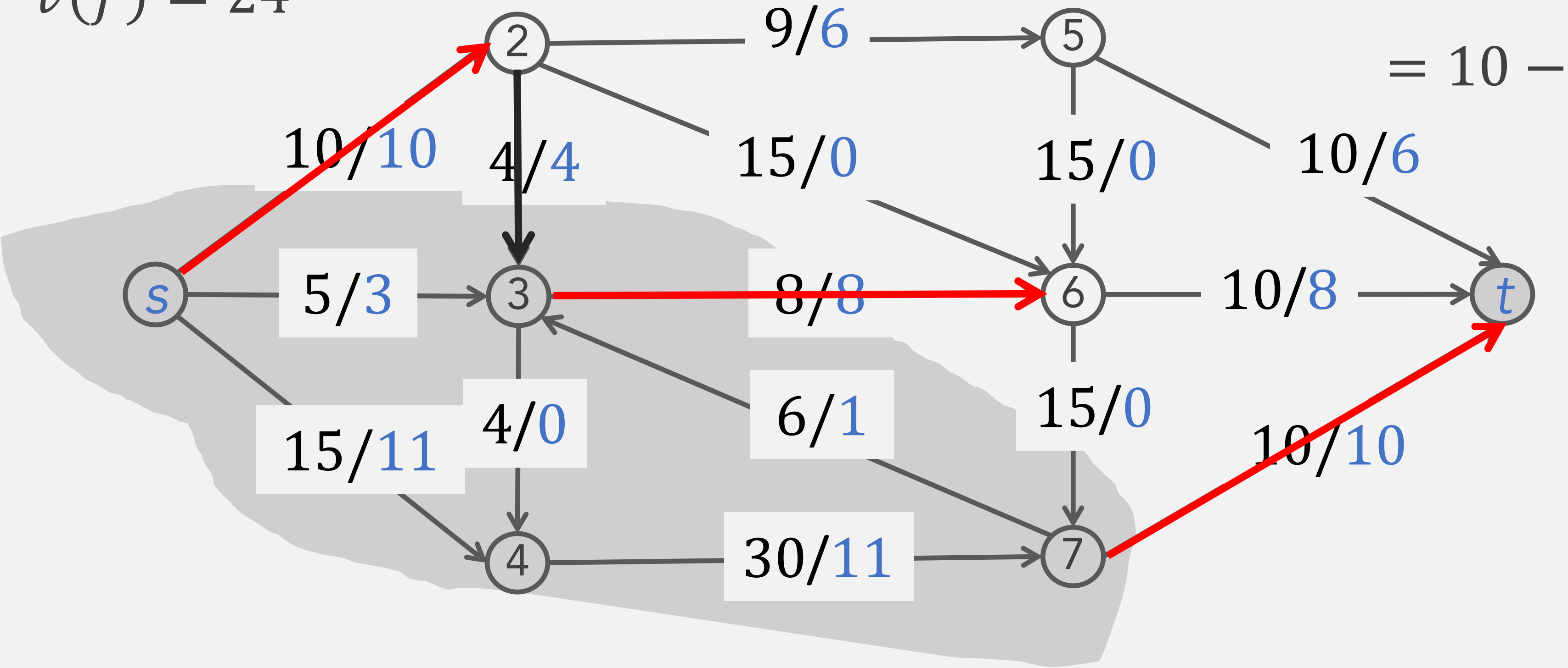
Flow value lemma

Flow-value lemma. Let f be any flow, and let (A, B) be any s–t cut. Then the net flow across the cut is equal to the amount leaving s (i.e., value of flow).

$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = v(f)$$

$v(f) = 24$

Net flow
$= 10 - 4 + 8 - 0 + 10 = 24$



Weak duality

Weak duality. Let f be any flow, and let (A, B) be any $s - t$ cut. Then the value of the flow is at most the capacity of the cut.

$$v(f) \leq cap(A, B)$$

Proof.

$$v(f) = \sum_{e \text{ outof } A} f(e) - \sum_{e \text{ into } A} f(e)$$

$$\leq \sum_{e \text{ outof } A} f(e)$$

$$\leq \sum_{e \text{ outof } A} c(e)$$

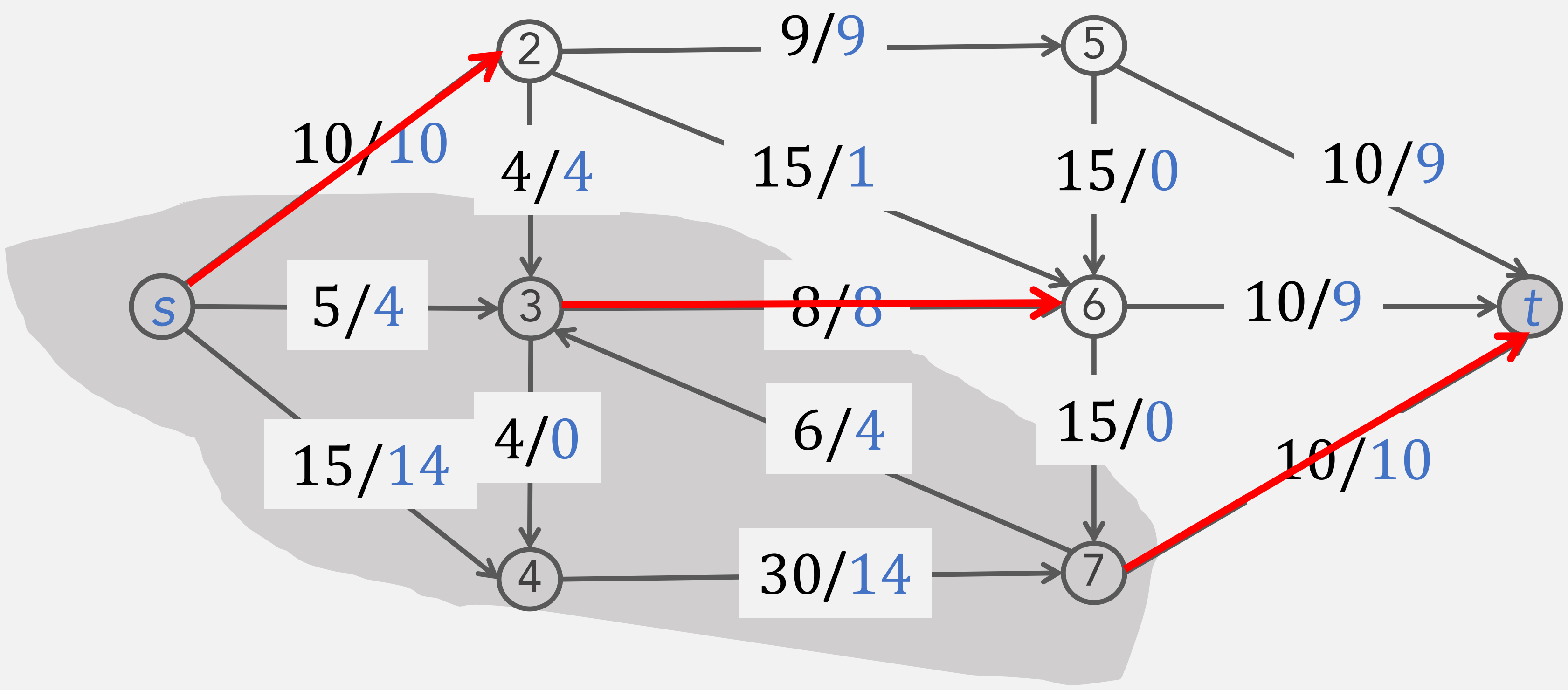
$$= cap(A, B)$$

When does it become **equality**?

- No flow coming into A
- Flows saturate outgoing edges

Weak duality ⇒ certificate of optimality

Corollary (of weak duality). Let f be any flow, and (A, B) be any s–t cut. If $v(f) = cap(A, B)$, then f is a max flow, and (A, B) a min cut.



Value of flow = 28

Cut capacity = 28

⇒ Value of flow ≤ 28

Max-flow min-cut theorem

Theorem. Value of max flow = capacity of min cut

Strong duality

MAXIMAL FLOW THROUGH A NETWORK

L. R. FORD, JR. AND D. R. FULKERSON

ON THE MAX FLOW MIN CUT THEOREM OF NETWORKS

G. B. Dantzig

D. R. Fulkerson

P-826

https://apps.dtic.mil/dtic/tr/fulltext/u2/605014.pdf

April 15, 1955

1956 IRE TRANSACTIONS ON INFORMATION THEORY

A Note on the Maximum Flow Through a Network*

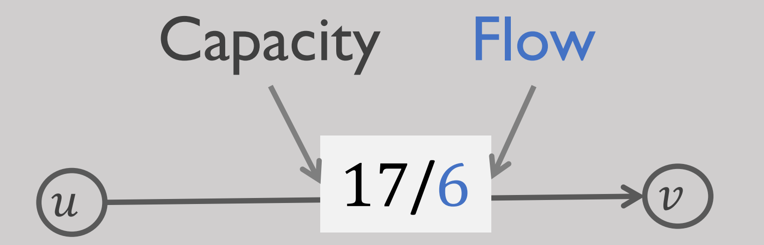
P. ELIAS†, A. FEINSTEIN‡, AND C. E. SHANNON§

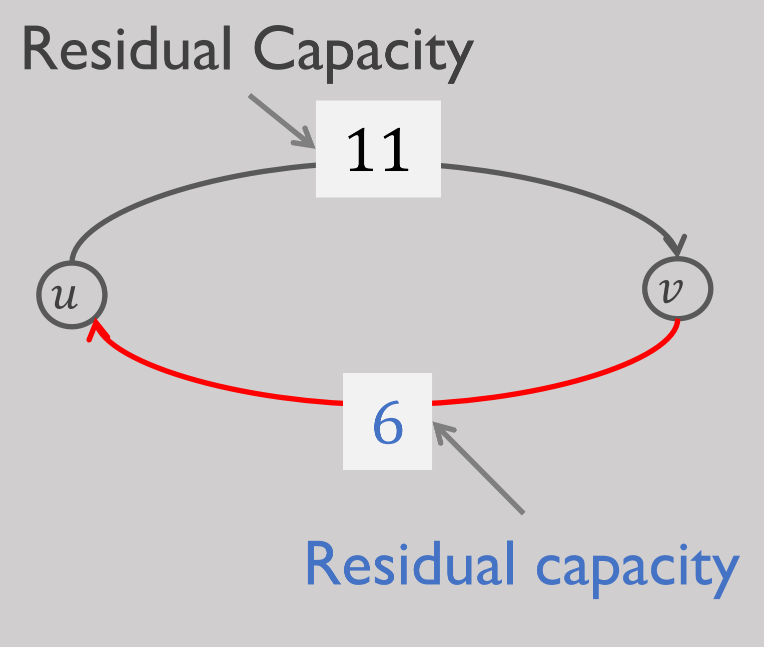
Residue graph

- Original edge: $e = (u, v) \in E$
  - Flow $f(e)$, capacity $c(e)$
- Residual edge: "Undo" flow sent
  - $e = (u, v)$ and $e^R = (v, u)$
  - Residual capacity

$$c_f(e) = \begin{cases} c(e) - f(e) & if\ e \in E \\ f(e) & if\ e^R \in E \end{cases}$$

- Residual graph $G_f = (V, E_f)$
  - Residual edges with positive residual capacity
  - $E_f = \{e: f(e) < c(e)\} \cup \{e^R: f(e) > 0\}$

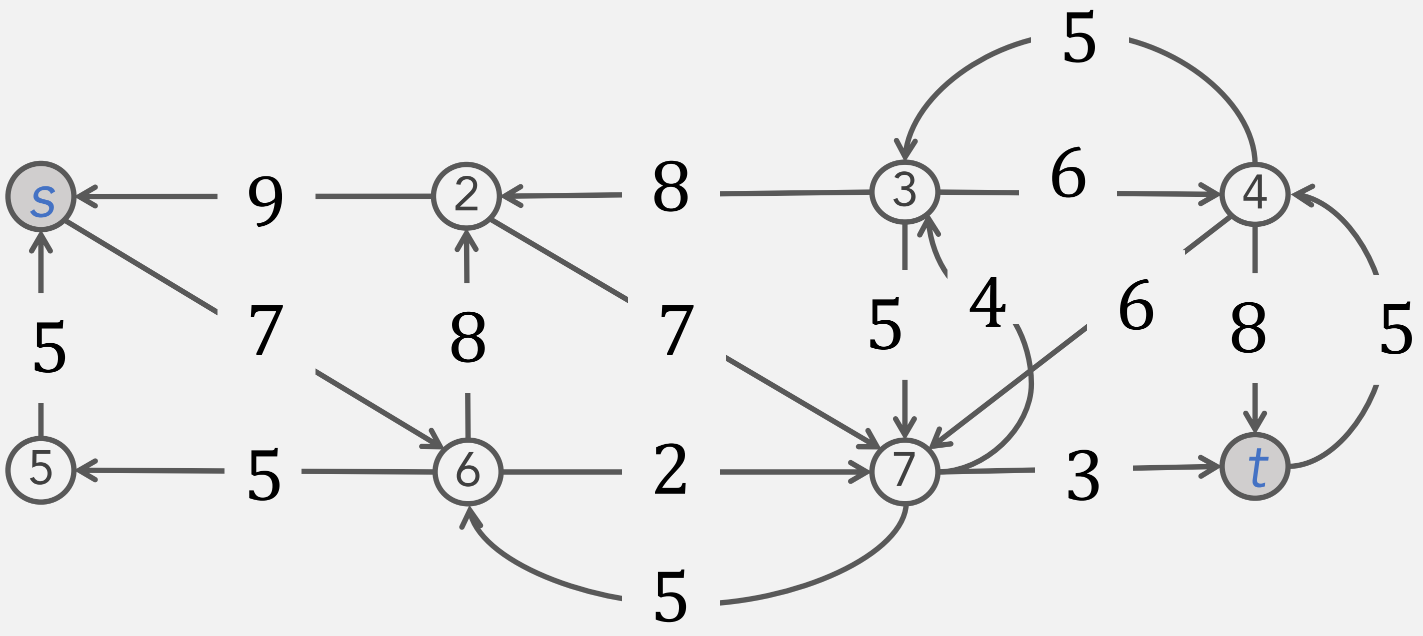




Augmenting path

- An augmenting path is a simple $s \rightsquigarrow t$ path in residual graph G_f.
- The bottleneck capacity of an augmenting path P is the minimum residual capacity of any edge in P.

Which is the augmenting path of highest bottleneck capacity?



Augmenting path theorem

Theorem. f is a max flow iff. no augmenting paths ($s \rightsquigarrow t$) in G_f.

"Algorithmic" max-flow min-cut thm

Proof. We show that the following are equivalent ($a \Rightarrow b \Rightarrow c \Rightarrow a$)

a. f is a max flow
b. There is no augmenting path (with respect to f)
c. There exists a cut (A, B) such that $cap(A, B) = v(f)$

Corollary of weak duality. [(A, B) also a min-cut]

a $\Leftrightarrow$ c

Max-flow min-cut theorem

Value of max flow = capacity of min cut

Augmenting path theorem: proof

a. f is a max flow
b. There is no augmenting path (with respect to f)
c. There exists a cut (A, B) such that $cap(A, B) = v(f)$

- a ⇒ b. We show contrapositive ¬b ⇒ $\neg a$

Lemma (augmented flow). Let P be an augmenting path with respect to f. Then f' below is a feasible flow with $v(f') > v(f)$.

$\delta \leftarrow$ bottleneck capacity of augmenting path P

For each $e \in P$, $f'(e) = \begin{cases} f(e) + \delta, & e \in E \\ f(e) - \delta, & e^R \in E \end{cases}$

Pf.

- Exercise. Verify f' is a feasible flow (i.e., capacity and conservation hold).
- $v(f') = v(f) + \delta > v(f)$ because only first edge in P leaves s.

Augmenting path theorem

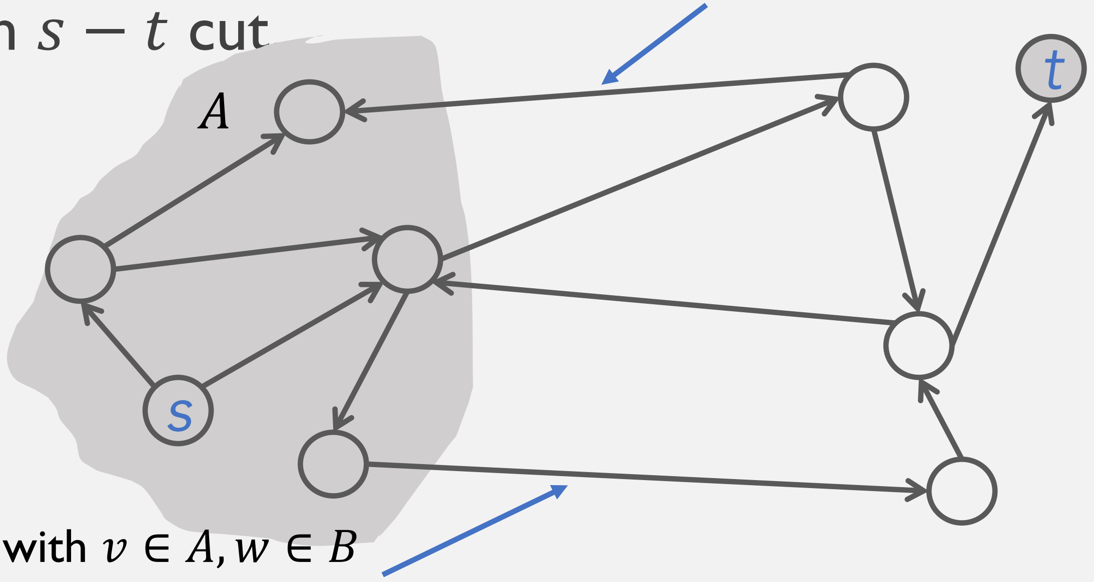
a.	f is a max flow
b.	There is no augmenting path (with respect to f)
c.	There exists a cut (A, B) such that $cap(A, B) = v(f)$

- b $\Rightarrow$ c. Assuming G_f has no augmenting path
  - Let A be the set of nodes reachable from s in G_f.
  - Clearly $s \in A$, and $t \notin A$. $(A, B = S \backslash A)$ an $s - t$ cut
  - Obs. On edges of G_f go from A to B.

$$v(f) = \sum_{e \text{ outof } A} f(e) - \sum_{e \text{ into } A} f(e)$$
$$= \sum_{e \text{ outof } A} c(e) - 0$$
$$= cap(A, B)$$

Original G

edge $e = (v, w)$ with $v \in B, w \in A$ must have $f(e) = 0$



edge $e = (v, w)$ with $v \in A, w \in B$ must have $f(e) = c(e)$